# Fourier analysis and holomorphic decomposition on the one-sheeted hyperboloid

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**Abstract**

We first prove a Cauchy-type integral representation for classes of functions holomorphic in four priviledged tuboid domains of the quadric $X^{(c)}$ in $\mathbb{C}^3$, defined as the complexification of the one-sheeted hyperboloid $X$ with equation $x_0^2 - x_1^2 - x_2^2 = -1$. From a physical viewpoint, this hyperboloid can be used for describing both the two-dimensional de Sitter and anti-de Sitter universes.

For two of these tuboids, called "the Lorentz tuboids" $\mathcal{T}^+$ and $\mathcal{T}^-$ and relevant for de Sitter Quantum Field Theory, the boundary values onto $X$ of functions holomorphic in these domains admit *continuous* Fourier-Helgason-type transforms $\tilde{f}_{\pm,\nu}(\xi)$, where $\nu$ labels the representations of the principal series of the group $SO_0(1,2)$ and $\xi$ belongs to the asymptotic future cone $C^+$ of $X$. Considering the case of functions invariant under a stabilizer subgroup $SO_0(1,1)$, the link of the previous transformation with the spherical Laplace transformation of invariant Volterra kernels is exhibited. For the other two tuboids, called the "chiral tuboids" $T_{\rightarrow}$ and $T_{\leftarrow}$ and relevant for anti-de Sitter Quantum Field Theory, the boundary values onto $X$ of functions holomorphic in these domains admit *discrete* Fourier-Helgason-type transforms $\tilde{f}_{\leftrightarrows,\ell}(\xi)$, where $\ell$ labels the representations of the discrete series of the group $SO_0(1,2)$ and $\xi$ varies in corresponding domains $C_{\leftrightarrows}$ of the complexified of $C^+$ in $\mathbb{C}^3$. In both cases, the inversion formulae for these transformations are derived by using the previous Cauchy representation for the respective classes of functions. The decomposition of functions on $X$ into sums of boundary values of holomorphic functions from the previous four tuboids gives a complete and explicit treatment of the Gelfand-Gindikin program for the one-sheeted hyperboloid.

## 1 Introduction

Although pertaining to the general approach to Fourier analysis on symmetric spaces $G/K$ whose framework and methods have been developped in particular by S. Helgason [1] [2], the case of harmonic analysis on the one-sheeted hyperboloid has necessitated a special treatment, due to the fact that it exhibits a situation (in fact, the simplest one) in which $K$ is a *non-compact* subgroup of $G$. This has been performed in particular in works by V. Molchanov [3] and J. Faraut [4]. In this article, we wish to give a new presentation of the Fourier-Helgason transformation on the one-sheeted hyperboloid $X$ which is more related to complex analysis on the corresponding complexified quadric $X^{(c)}$ with equation $z^2 \equiv z_0^2 - z_1^2 - z_2^2 \ = \ -1$ in $\mathbb{C}^3$ (considered as a homogeneous space of the complex Lorentz group $SO_0(1,2)^{(c)}$); in fact, the results of this study will closely parallel those of the usual Fourier analysis on $\mathbb{R}^2$ in connection with complex analysis on $\mathbb{C}^2$.

Our starting point is the existence of four distinguished holomorphy domains in $X^{(c)}$ which we call "tuboids" for the following reasons:

i) there exists a global holomorphic representation of $X^{(c)}$ in $\mathbb{C}^2$ which displays a correspondence between these four domains and the tubes defined as the products of upper and lower half-planes in these two complex variables.

ii) each of these domains is bordered by the whole set of real points of $X^{(c)}$ (i.e. $X$), and is locally a tuboid in the sense of [5] (see also the Appendix of [6]): the notion of boundary values of holomorphic functions from these domains on the reals is thus well-defined (in the sense of functions or of distributions).

These four domains are invariant under the action of the *real* group $SO_0(1,2)$ on $X^{(c)}$. The first two domains $\mathcal{T}^+$ and $\mathcal{T}^-$ play the same role as the Lorentz tubes $T^\pm = \mathbb{R}^3 + iV^\pm$ in $\mathbb{C}^3$ ($V^+$ and $V^-$ denoting the open future and past cones in $\mathbb{R}^3$); they will be called below "Lorentz tuboids of $X^{(c)}$". The other two domains $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$ which are not simply-connected and are distinguished from each other by an orientation prescription will be called "chiral tuboids of $X^{(c)}$".

We shall consider classes of holomorphic functions in these four tuboids, sufficiently regular at infinity so as to admit a Cauchy-type integral representation in terms of their boundary values on $X$. The corresponding Cauchy kernel will be seen to be proportional to the inverse of the Minkowskian quadratic form $(z - z')^2$ restricted to $X^{(c)} \times X^{(c)}$; note that this kernel is invariant under the action of the complexified group $SO_0(1,2)^{(c)}$ of $SO_0(1,2)$ on $X^{(c)}$.

Our purpose is to provide *a theory of the Fourier-Helgason (FH) transformation specifically adapted to the classes of functions on $X$ which are boundary values of holomorphic functions from either one of these four tuboids.* While this paper is purely mathematical, its results have a natural physical interpretation in the context of de Sitter and anti-de Sitter Quantum Field Theory, where the relevant correlation functions belong precisely to such classes [6, 7, 8, 9].

Introducing the bilinear form $[z \cdot z'] = z_0 z_0' - z_1 z_1' - z_2 z_2'$ on $\mathbb{C}^3$, we shall make use of the FH-kernel $[z \cdot \xi]^s$ for various types of configurations of the pair $(z, \xi)$, with $z^2 \equiv [z \cdot z] = -1$ and $\xi^2 \equiv [\xi \cdot \xi] = 0$.

In the case of holomorphic functions in the Lorentz tuboids (already considered in [6]), one will take advantage of the fact that the complex number $[z \cdot \xi]$ remains in the upper (resp. lower) half-plane when $z$ varies in $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) and $\xi$ lies in the asymptotic future cone $C^+ = \partial V^+$ of $X$. One then makes use of the following form of the FH-kernel $[x_\pm \cdot \xi]^s = \lim_{z \in \mathcal{T}^\pm, \mathrm{Im} z \to 0} [z \cdot \xi]^s$ in place of the Gelfand-type forms $|[x \cdot \xi]|^s$ and $\mathrm{sgn}([x \cdot \xi])|[x \cdot \xi]|^s$ currently used for introducing the FH-transformation on $X$ (see e.g. [4] [3] and references therein). In fact, it will be seen that for the functions $f(x)$ which are boundary values of a holomorphic function respectively from $\mathcal{T}^+$ or $\mathcal{T}^-$ our prescription for $[x \cdot \xi]^s$ selects a *unique non-vanishing* FH-transform, denoted respectively $\tilde{f}_{-,\nu}(\xi)$ or $\tilde{f}_{+,\nu}(\xi)$, having restricted $s$ to vary in the range of values $s = -1/2 - i\nu$ which label the principal series of irreducible unitary representations of the group $SO_0(1,2)$. The existence of such a unique relevant FH-type transform is the analogue of *the support property of the Fourier transforms* for the functions (or tempered distributions) in $\mathbb{R}^3$ which are boundary values of holomorphic functions in either tubes $T^+$ or $T^-$ of $\mathbb{C}^3$: such functions are characterized by the fact that their Fourier transforms have their support contained in the closure of

either one of the cones $V^\pm$ (see [10], chapter 8).

In order to invert this FH-transformation, we shall also make use of the analyticity properties in $z$ of the inverse FH-kernel $[z \cdot \xi]^{-1/2+i\nu}$ which allow one to define the expected inverse as a holomorphic function $F(z)$ in the relevant Lorentz tuboid $\mathcal{T}^\pm$ of $X^{(c)}$. The proof that the initially given function $f(x)$ is indeed the boundary value of $F(z)$ from its tuboid is obtained by directly computing the composition of the direct and inverse FH-kernels and showing that it yields explicitly the Cauchy-type representation in $\mathcal{T}^\pm$ established at first. This generalizes the procedure according to which the standard Cauchy kernel $(z-x)^{-1}$ emerges from the Fourier inversion computation *with support properties* , as being equal (for $z$ in the upper half-plane) to the integral $1/i \int_0^\infty e^{i(z-x)t}\, dt$.

Among the functions or distributions $f(x)$ on $X$ which are boundary values of holomorphic functions from $\mathcal{T}^+$ or $\mathcal{T}^-$ it is interesting to consider those which are moreover invariant under a subgroup of $SO_0(1,2)$ such as the stabilizer (isomorphic to $SO_0(1,1)$) of a given "base point" $b$ of $X$ (chosen below as $b=(0,0,1)$). In fact, such functions or distributions can be identified with invariant kernels on $X$ which are boundary values of holomorphic functions defined in the "cut-domain" $\{(z,z') \in X^{(c)} \times X^{(c)};\ (z-z')^2 \in \mathbb{C}\backslash\mathbb{R}^+\}$. These objects have been studied under the name of *perikernels* in [11], [12] and their discontinuities on the cut $\{(z,z');\ (z-z')^2 \geq 0\}$ define *Volterra kernels* in the sense of [14]. By applying the previous Fourier-Helgason transformation to this class of $SO_0(1,1)$-invariant functions and to their discontinuities, we shall then reobtain the theory of the *"spherical Laplace transformation" of invariant Volterra kernels* of [13], [14].

In the case of holomorphic functions in the chiral tuboids, the definition of the FH-transformation will be qualitatively different, since the variable $\xi$ may now vary in two corresponding domains of the complexified cone $C^{(c)}$ of $C^+$, while the exponent $s$ is restricted to the set of integral values $s = -\ell - 1$, where $\ell \geq 0$ labels the discrete series of irreducible unitary representations of $SO_0(1,2)$. This discretization, which is due to the topological equivalence of $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$ with the product of $S_1$ with wedges in $\mathbb{R}^3$, would be lifted if these chiral tubes were replaced by their universal coverings (i.e. $S_1$ by $\mathbb{R}$), and $X$ and its symmetry group $SO_0(1,2)$ by their corresponding coverings [8].

After having introduced the geometry of the four tuboids $\mathcal{T}^+$, $\mathcal{T}^-$, $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$ of the complex quadric $X^{(c)}$ in our section 2, we shall devote section 3 to the proof of a Cauchy-type representation for functions holomorphic in these four domains.

A "Lorentzian" FH-transformation is then introduced and studied in section 4 .For the functions on $X$ which are boundary values of holomorphic functions in the tuboids $\mathcal{T}^\pm$, the inversion of this FH-transformation is then performed by the Cauchy kernel method, as explained above.

The case of $SO_0(1,1)-$invariant functions (i.e. of invariant perikernels) and its connection with the theory of spherical Laplace transformation is treated in section 5.

Section 6 is devoted to the "chiral" FH-transformation and to the corresponding representation of the functions on $X$ which are boundary values of holomorphic functions in the tuboids $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$.

In the final section 7, we shall summarize our results concerning the decomposition of functions on $X$ as sums of boundary values of holomorphic functions from the four tuboids $\mathcal{T}^+$, $\mathcal{T}^-$, $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$ of $X^{(c)}$. Our characterization of this decomposition in terms of *Lorentzian and chiral FH-transforms of the four components* gives a complete and explicit treatment of the *Gelfand-Gindikin program* (see [15], [16] and references therein) for the case of the one-sheeted hyperboloid in dimension 2.

Extensions of our results in two directions will be given in forthcoming papers: on the one hand, results concerning holomorphic functions in the Lorentz tuboids can be generalized to the case of the complexified $n-$dimensional one-sheeted hyperboloid in $\mathbb{C}^{n+1}$; on the other hand, results concerning holomorphic functions in chiral tuboids can be generalized to the case of the complexified $n-$dimensional quadric $[z \cdot z] = 1$, where $[z \cdot z]$ denotes a quadratic form with signature $(+,+,-,\cdots,-)$. These two cases have respective applications to Quantum Field Theory in de Sitter and anti-de Sitter spacetime manifolds in dimension $n$ (see [6, 7, 8, 9]).

## 2 The four basic tuboids of the complexified one-sheeted hyperboloid

The space $\mathbb{R}^3$ of the variables $x = (x_0, x_1, x_2)$ is equipped with the Minkowskian bilinear form

$$[x \cdot y] = x_0 y_0 - x_1 y_1 - x_2 y_2, \qquad (x^2 = [x \cdot x]). \tag{1}$$

We introduce the "light-cone"

$$C = \left\{ x \in \mathbb{R}^3 : \; x^2 = 0; \; x \neq 0 \right\}, \tag{2}$$

the (closed) "future cone"

$$\overline{V}^+ = \left\{ x \in \mathbb{R}^3 : \; x^2 \geq 0, \; x_0 \geq 0 \right\} \tag{3}$$

its interior $V^+$ and its boundary $C^+ = C \cap \overline{V}^+$. One defines similarly $V^- = -V^+$, etc...

The one-sheeted hyperboloid $X = \left\{ x \in \mathbb{R}^3 : x^2 = -1 \right\}$ is equipped with a *"causal" ordering relation* induced by that of the ambient Minkowskian space $\mathbb{R}^3$, namely $\forall (x, y) \in X \times X, \;\; x \geq y \;\leftrightarrow\; x - y \in \overline{V}^+$. The "future cone" [1] of a given point $x$ in $X$ is $\Gamma^+(x) = \{ y \in X : y \geq x \}$. Analogously one defines $\Gamma^-(x)$. The "light-cone"$\partial\Gamma(x)$ of $x$ on $X$, namely the boundary set of $\Gamma^+(x) \cup \Gamma^-(x)$, is the pair of linear generatrices of $X$ containing the point $x$.

The invariance group of $X$ is the pseudo-orthogonal group $SO(1, 2)$ leaving invariant the bilinear form (1) and we shall call "Lorentz group $G$" the connected component $SO_0(1, 2)$ of the latter. We choose the integration measure $\mathrm{d}\sigma(x)$ on $X$ as associated with the $G$-invariant volume form $\left.\frac{\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2}{\mathrm{d}(x^2+1)}\right|_X$ (in Leray's notations [17]), namely:

$$\mathrm{d}\sigma(x) = \frac{\mathrm{d}x_1 \mathrm{d}x_2}{2|x_0|} \tag{4}$$

Since the group $G$ acts in a transitive way on $X$, it is convenient to distinguish a *base point* $b$ in $X$ which we choose to be $b = (0, 0, 1)$.

The complexified hyperboloid $X^{(c)} = \{ z = x + iy \in \mathbb{C}^3 : z^2 = \; -1 \}$ is equivalently described as the set

$$X^{(c)} = \{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : \; x^2 - y^2 = -1, \; [x \cdot y] = 0 \}. \tag{5}$$

The set of complex points of $X^{(c)}$ can be decomposed as the union of the following three disjoint sets denoted by $\mathcal{T}_L$, $\mathcal{T}_0$ and $\mathcal{T}_\chi$, separately invariant under the action of $G$ (defined on $X^{(c)}$ by $gz = gx + igy$ for all $g \in G$):

i)

$$\mathcal{T}_0 = \{ z = x + iy \in X^{(c)}; \;\; y^2 = x^2 + 1 = 0 \} \tag{6}$$

is the set of all complex points of the straight lines which generate $X$;

ii)

$$\mathcal{T}_L = \{ z = x + iy \in X^{(c)}; \;\; y^2 = x^2 + 1 > 0 \} \tag{7}$$

In view of the equation $[x \cdot y] = 0$ (5), all the points in $\mathcal{T}_L$ are also such that $-1 < x^2 < 0$ and correspondingly $0 < y^2 < 1$.

iii)

$$\mathcal{T}_\chi = \{ z = x + iy \in X^{(c)}; \;\; y^2 = x^2 + 1 < 0 \} \tag{8}$$

These three sets can also be characterized in terms of the three families of planes $\Pi$ containing the origin and respectively tangent, transverse, or exterior to the light-cone $C$. While $\mathcal{T}_0$ is the union of complex points of sections of $X$ by planes $\Pi$ tangent to $C$, $\mathcal{T}_L$ appears as the union of complex points $z$ of all (hyperbolic) sections of $X$ by planes $\Pi$ transverse to $C$ and $\mathcal{T}_\chi$ as the union of complex points of all (elliptic) sections of $X$ by planes $\Pi$ exterior to $C$. In fact, each complex point $z = x + iy$ of $X^{(c)}$ determines a unique plane $\Pi = \Pi(z)$ containing $x$ and $y$, whose complexified contains $z$ and $\overline{z}$, and which

[1] This terminology is justified by the interpretation of the quadric $X$ as a two-dimensional de Sitter spacetime.

belongs to either one of the three previous categories according to whether $y^2$ is equal to zero, positive or negative: this is because the two orthogonal vectors $x$ and $y$ (in the sense of the Minkowskian scalar product) indicate the type of metric obtained on the plane $\Pi$, which is respectively, in view of the signs of $x^2$ and $y^2$, either degenerate or Minkowskian or Euclidean.

### 2.1 The set $\mathcal{T}_L$ and the tuboids $\mathcal{T}^+$ and $\mathcal{T}^-$

We define the Lorentz invariant sets $\mathcal{T}^+$ and $\mathcal{T}^-$ as the connected components of $\mathcal{T}_L$, specified by adding the respective conditions $y_0 > 0$ and $y_0 < 0$ to the definition of $\mathcal{T}_L$.

These two sets, which we call "Lorentz tuboids" are complex conjugate of each other and an equivalent definition of them is:

$$\mathcal{T}^+ = \mathrm{T}^+ \cap X^{(c)}, \qquad \mathcal{T}^- = \mathrm{T}^- \cap X^{(c)}, \tag{9}$$

where $\mathrm{T}^\pm = \mathbb{R}^3 + iV^\pm$ are the Lorentz tubes in the ambient space $\mathbb{C}^3$.

We shall now give two alternative characterizations of $\mathcal{T}^\pm$:

**Proposition 1** *$\mathcal{T}^+$ (resp. $\mathcal{T}^-$) is the set of all points $z = x + iy$ in $X^{(c)}$ such that the inequality* $\mathrm{Im}([z\cdot\xi]) > 0$ *(resp. $< 0$) holds for* all $\xi \in C^+$.

The proof is immediate since the condition $\forall\xi \in C^+,\ [y\cdot\xi] > 0$ (resp. $< 0$) is equivalent to the condition $y \in V^+$ (resp. $V^-$).

**Proposition 2** *The domains $\mathcal{T}^+$ and $\mathcal{T}^-$ are generated respectively by the action of the Lorentz group $G$ on the following one-dimensional subsets (namely half-circles) of $X^{(c)}$:*

$$\gamma^+ = \ \{z = x + iy \in X^{(c)};\, z = z_u = (i\sin u, 0, \cos u),\ \ 0 < u < \pi\} \tag{10}$$

*and*

$$\gamma^- = \ \{z = x + iy \in X^{(c)};\, z = z_u = (i\sin u, 0, \cos u),\ \ -\pi < u < 0\} \tag{11}$$

**Proof.** It is sufficient to consider the case of $\mathcal{T}^+$ and $\gamma^+$. Let $G_b$ be the stabilizer of $b$ in G. The action of the one-parameter subgroup $G_b$ on $\gamma^+$ generates the set

$$G_b\ \gamma^+ = \{z \in X^{(c)};\, z = (i\sin u\cosh\alpha,\ i\sin u\sinh\alpha,\ \cos u),\ \ 0 < u < \pi\} \tag{12}$$

Let now $z = x+iy$ be an arbitrary point in $\mathcal{T}^+$; in view of (7), $x$ belongs to an hyperboloid of the form $x^2 = -\cos^2 u$, and therefore (by transitivity) there exists a $g \in G$ such that $gx = b_u = \cos u\ b$. Then the point $gy$ must be such that $[gy\cdot b_u] = [y\cdot x] = 0$ with $(gy)^2 = \sin^2 u$ and $(gy)_0 > 0$, which implies that $gz = b_u + igy$ is of the form (12); therefore one has $gz \in G_b\ \gamma^+$ and $z \in G\gamma^+$. ■

The domain $\mathcal{T}^+$ is a tuboid over $X$ in $X^{(c)}$ whose profile at the base point $b$ is the cone $V^+(b) = \{y \in \mathbb{R}^3; y_0 > |y_1|,\ y_2 = 0\}$; the latter is in fact obtained by taking the limit $u \to 0$ from the set $G_b\ \gamma^+$ into the tangent plane to $X$ at $b$. The domain $\mathcal{T}^-$ is similarly described by replacing the condition $0 < u < \pi$ by $-\pi < u < 0$ in the previous analysis: it is a tuboid whose profile at the point $b$ is the cone $V^-(b) = -V^+(b)$.

The following property holds:

**Proposition 3** *The projection of the domain $\mathcal{T}^+$ (or $\mathcal{T}^-$) in the complex plane of the coordinate $z_2 = -[z\cdot b]$ is the cut-plane $\Theta_L = \mathbb{C}\setminus\{[-\infty,-1]\cup[1,+\infty]\}$. The image of the domain $\mathcal{T}^-\times\mathcal{T}^+$ (or $\mathcal{T}^+\times\mathcal{T}^-$) by the mapping $(z,z') \to [z\cdot z']$ is the cut-plane $\hat{\Theta}_L = \mathbb{C}\setminus[-\infty,-1]$.*

**Proof.** a) We first notice that the set $\mathcal{T}_0^+ = \{z = (i\sin(u+iv),\ 0,\ \cos(u+iv)); 0 < u < \pi, v \in \mathbb{R}\}$ is contained in $\mathcal{T}^+$, since all these points $z = x+iy$ are such that $y_0 = \sin u\cosh v > 0$ and $y^2 = \sin^2 u > 0$. One then checks that the corresponding range of the projection $\{z_2 = \cos(u+iv);\ 0 < u < \pi,\ v \in \mathbb{R}\}$ is already $\Theta_L$.

Let us now show that the points exterior to $\Theta_L$, namely the points $z_2$ which are real and such that $|z_2| \geq 1$ cannot be the projections of points $z$ in $\mathcal{T}^+$ or in $\mathcal{T}^-$. In fact, all points $z = (z_0, z_1, z_2) \in X^{(c)}$ with $z_2^2 > 1$ belong to a complex hyperbola $h(z_2)$ with equation $\hat{z}^2 \equiv z_0^2 - z_1^2 = z_2^2 - 1 > 0$. Keeping the same notations for the two-dimensional Minkowskian bilinear form, we can say that all the complex points $z = (\hat{z}, z_2)$, with $\hat{z} = (z_0, z_1) = \hat{x} + i\hat{y}$ in $h(z_2)$ satisfy the conditions $\hat{x}^2 - \hat{y}^2 > 0$ and $[\hat{x} \cdot \hat{y}] = 0$; the latter imply $\hat{x}^2 > 0$ and $\hat{y}^2 \equiv y^2 < 0$ and therefore $z \notin \mathcal{T}_L$. As for the points $z$ such that $z_2^2 = 1$, they are such that $\hat{x}^2 = y^2 = 0$ and therefore also not in $\mathcal{T}_L$.

b) We first check that the image of the subset $\left\{(z', z)\,;z' \in \gamma^-, z \in \mathcal{T}_0^+\right\}$ of $\mathcal{T}^- \times \mathcal{T}^+$ in the plane of the variable $[z \cdot z']$ is already $\hat{\Theta}_L$. Let in fact: $z' = (-i \sin u', 0, \cos u')$, with $0 < u' < \pi$, and $z \in \mathcal{T}_0^+$ as described in a). We then have: $[z \cdot z'] = -\cos(u + u' + iv)$ with $u + u', v$ varying in the range $0 < u + u' < 2\pi$; $v \in \mathbb{R}$, and therefore the corresponding range of $[z \cdot z']$ is $\hat{\Theta}_L$.

It remains to show that one cannot have $[z \cdot z']$ real and $\leq -1$, or equivalently $(z - z')^2 \geq 0$, if $z$ belongs to $\mathcal{T}^+$ and $z'$ to $\mathcal{T}^-$. This follows from a simple argument in the ambient space $\mathbb{C}^3$: the conditions $z \in \mathcal{T}^+$ and $z' \in \mathcal{T}^-$ imply that the vector $Z = z - z' = X + iY$ belongs to the tube $T^+$, so that one has $Y^2 > 0, Y_0 > 0$. But the condition $Z^2 \geq 0$, which is equivalent to the pair $X^2 - Y^2 \geq 0,\ [X \cdot Y] = 0$ cannot be implemented with $Y^2 > 0$, since the latter implies $X^2 \leq 0$.

It is clear that at each step of this proof, the roles of the tubes $\mathcal{T}^+$ and $\mathcal{T}^-$ can be inverted without changes in the conclusions. ■

## 2.2 The set $\mathcal{T}_\chi$ and the tuboids $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$

We shall also use later the following alternative definition of $\mathcal{T}_\chi$:

**Proposition 4** *$\mathcal{T}_\chi$ is the set of all complex points $z = x + iy$ in $X^{(c)}$ such that there exist two distinct vectors $\xi_+$ and $\xi_-$ in $C^+$ (depending on $y$) satisfying the conditions:* $\mathrm{Im}([z \cdot \xi_+]) = \mathrm{Im}([z \cdot \xi_-]) = 0$.

**Proof.** The stated conditions are equivalent to the existence of two distinct planes tangent to the cone $C^+$ and intersecting each other along the support of $y$. The latter condition is of course equivalent to the fact that $y$ is outside the union of $V^+$ and $V^-$, namely that $y^2$ is negative. ■

Let $e$ be any vector in the cone $V^+$; for every $z = x + iy$ in $\mathcal{T}_\chi$, we put $\epsilon(z) = \mathrm{sgn}\ \mathrm{Det}(e, x, y)$, this sign being independent of the choice of $e$ since the plane $\Pi(z)$ is exterior to $C$. It is then clear that for every $g \in G$, one has $\epsilon(gz) = \epsilon(z)$, so that the following two sets are Lorentz-invariant.

$$\mathcal{T}_\rightarrow = \{z = x + iy \in X^{(c)};\ y^2 < 0,\ \epsilon(z) = -\}, \tag{13}$$

$$\mathcal{T}_\leftarrow = \{z = x + iy \in X^{(c)};\ y^2 < 0,\ \epsilon(z) = +\}. \tag{14}$$

These two connected components of $\mathcal{T}_\chi$, which we call "chiral tuboids", are complex conjugate of each other.

**Proposition 5** *The domains $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$ are generated respectively by the action of the Lorentz group $G$ on the following one-dimensional subsets (namely half-branches of hyperbola) of $X^{(c)}$:*

$$h_\rightarrow = \ \{z = x + iy \in X^{(c)};\ z = z_v = (0, i \sinh v, \cosh v),\ \ v > 0\} \tag{15}$$

*and*

$$h_\leftarrow = \ \{z = x + iy \in X^{(c)};\ z = z_v = (0, i \sinh v, \cosh v),\ \ v < 0\} \tag{16}$$

**Proof.** It is sufficient to consider the case of $\mathcal{T}_\rightarrow$ and $h_\rightarrow$. The action of the one-parameter subgroup $G_b$ on $h_\rightarrow$ generates the set

$$G_b\ h_\rightarrow = \{z \in X^{(c)};\ z = (i \sinh v \sinh \alpha,\ i \sinh v \cosh \alpha,\ \cosh v),\ \ v > 0\} \tag{17}$$

Let now $z = x + iy$ be an arbitrary point in $\mathcal{T}_\rightarrow$; in view of (8), $x$ belongs to an hyperboloid of the form $x^2 = -\cosh^2 v$, and therefore (by transitivity) there exists a $g \in G$ such that $gx = b_v = \cosh v\ b$. Then

the point $gy$ must be such that $[gy \cdot b_v] = [y \cdot x] = 0$ with $(gy)^2 = -\sinh^2 v$ and $\epsilon(b_v + igy) = -$, which implies that $gz = b_v + igy$ is of the form (17); therefore one has $gz \in G_b h_\rightarrow$ and $z \in G h_\rightarrow$. ■

The tube $\mathcal{T}_\rightarrow$ is a tuboid over $X$ in $X^{(c)}$ whose profile at the base point $b$ is the cone $V_\rightarrow(b) = \{y \in \mathbb{R}^3; y_1 > |y_0|,\ y_2 = 0\}$; the latter is in fact obtained by taking the limit $v \to 0$ from the set $G_b h_\rightarrow$ into the tangent plane to $X$ at $b$. The tube $\mathcal{T}_\leftarrow$ is similarly described by replacing the condition $v > 0$ by $v < 0$ in the previous analysis: it is a tuboid whose profile at the point $b$ is the cone $V_\leftarrow(b) = -V_\rightarrow(b)$.

**A simple parametrization of $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$.**

We now give a parametrization of $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$ which exhibits these domains as being *semi-tubes* in $\mathbb{C}^2$. Let us parametrize $X^{(c)}$ as follows:

$$z = z[\theta, \Psi] \equiv (\sinh \Psi,\ \cosh \Psi \sin \theta,\ \cosh \Psi \cos \theta) \tag{18}$$

with $(\theta = u + iv, \Psi = \psi + i\varphi) \in \mathbb{C}^2/(2\pi\mathbb{Z})^2$. In this parametrization, the translations $u \to u + a$ represent the rotations with axis $Oz_0$, which leave the domains $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$ invariant. One thus expects that the latter are represented by semi-tubes in $\mathbb{C}^2$ bordered by surfaces $v = v_\pm(\psi, \varphi)$ in $(\theta, \Psi)-$space. By taking the imaginary parts in Eq.(18), one rewrites the defining condition of $\mathcal{T}_\chi$ as follows:

$$y^2 = \sin^2 \varphi - \sinh^2 v(\cosh^2 \psi - \sin^2 \varphi) \quad < 0, \tag{19}$$

which implies the following representation for $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$:

$$\mathcal{T}_\rightarrow \qquad \tanh v > \frac{|\sin \varphi|}{\cosh \psi} \tag{20}$$

$$\mathcal{T}_\leftarrow \qquad \tanh v < -\frac{|\sin \varphi|}{\cosh \psi} \tag{21}$$

We shall now prove the following property which is the analogue of Proposition 3:

**Proposition 6** *The projection of the domain $\mathcal{T}_\rightarrow$ (or $\mathcal{T}_\leftarrow$) in the complex plane of the coordinate $z_2 = -[z \cdot b]$ is the cut-plane $\Theta_\chi = \mathbb{C} \setminus [-1, 1]$. The image of the domain $\mathcal{T}_\leftarrow \times \mathcal{T}_\rightarrow$ (or $\mathcal{T}_\rightarrow \times \mathcal{T}_\leftarrow$) by the mapping $(z, z') \to [z \cdot z']$ is the cut-plane $\Theta_\chi$.*

**Proof.** a) In view of Eq. (18), we have:

$$z_2 = \cosh(\psi + i\varphi) \cos(u + iv) \tag{22}$$

Eq. (20) shows that $\mathcal{T}_\rightarrow$ contains the set parametrized by $\{(\theta, \Psi);\ \theta = u + iv;\ \ u \in \mathbb{R},\ v > 0,\ \Psi = \psi + i\varphi = 0\}$, whose image in the $z_2-$plane is (in view of (22)) exactly $\Theta_\chi$.

Let us now show that the points exterior to $\Theta_\chi$, namely the points $z_2 \in [-1, +1]$ cannot be the projections of points $z$ in $\mathcal{T}_\rightarrow$ or in $\mathcal{T}_\leftarrow$. In fact, all points $z = (z_0, z_1, z_2) \in X^{(c)}$ with $z_2^2 < 1$ belong to a complex hyperbola $h(z_2)$ with equation $\hat{z}^2 \equiv z_0^2 - z_1^2 = -(1 - z_2^2) < 0$. All the complex points $z = (\hat{z}, z_2)$, with $\hat{z} = (z_0, z_1) = \hat{x} + i\hat{y}$ in $h(z_2)$ satisfy the conditions $\hat{x}^2 - \hat{y}^2 < 0$ and $[\hat{x} \cdot \hat{y}] = 0$, which imply $\hat{x}^2 < 0$ and $\hat{y}^2 \equiv y^2 > 0$ and therefore $z \notin \mathcal{T}_\chi$. As for the points $z$ such that $z_2^2 = 1$, they are such that $\hat{x}^2 = y^2 = 0$ and therefore also not in $\mathcal{T}_\chi$.

b) In view of Proposition 5, it is sufficient to check that the image of the domain $\{(z', z)\,;\, z' \in h_\leftarrow, z \in \mathcal{T}_\rightarrow\}$ into the plane of the variable $[z \cdot z']$ is $\Theta_\chi$. Let therefore

$$z' = (0, -i \sinh v', \cosh v'), \quad \text{with} \quad v' > 0, \tag{23}$$

while $z$ is parametrized as in Eq.(18), the inequality (20) being satisfied.

Let $g_{v'}$ be the complex rotation with axis $Oz_0$ and angle $iv'$ whose effect is to change $z'$ into $b$. The action of this rotation on $z$ gives:

$$g_{v'}z = (\sinh\Psi,\ \cosh\Psi \sin(u + i(v + v')))\,,\ \cosh\Psi\cos(u + i(v + v')) \tag{24}$$

and one thus has:

$$[b \cdot g_{v'}z] = [z' \cdot z] = \cosh\Psi\cos(u + i(v + v'))\,. \tag{25}$$

Since $v' > 0$, it follows that:

$$\tanh(v + v') > \tanh v > \frac{|\sin\varphi|}{\cosh\psi}, \tag{26}$$

which shows that $g_{v'}z \in \mathcal{T}_{\rightarrow}$ and in view of a), that $[z \cdot z'] \in \Theta_{\chi}$. Of course all the points of $\Theta_{\chi}$ are obtained in this image, since (in view of (24),(26)), $g_{v'}z$ varies in the whole set $\mathcal{T}_{\rightarrow}$ when $z$ varies in the latter and $v'$ takes all positive values. ■

# 3 Cauchy-type representation in the tuboids $\mathcal{T}^+, \mathcal{T}^-, \mathcal{T}_{\rightarrow}, \mathcal{T}_{\leftarrow}$ and holomorphic decomposition on the one-sheeted hyperboloid

## 3.1 A global parametrization of the four tuboids

It is convenient to use the following parametrization of $\hat{X}^{(c)} = \{z \in X^{(c)};\ z_0 + z_1 \neq 0\}$:

$$\begin{gathered} z = z(\lambda,\mu): \\ z_0 = \frac{1+\lambda\mu}{\lambda-\mu},\quad z_1 = \frac{1-\lambda\mu}{\lambda-\mu},\quad z_2 = \frac{\lambda+\mu}{\lambda-\mu} \\ \text{with}\ \ (\lambda,\mu) \in \mathbb{C}^2 \setminus \delta, \end{gathered} \tag{27}$$

$\delta$ being the diagonal ($\lambda = \mu$), which represents points at infinity of $X^{(c)}$. The holomorphic compactification $\mathbb{S}_2 \times \mathbb{S}_2$ of $\mathbb{C}^2 \setminus \delta$ thus provides a corresponding compactification of $X^{(c)}$ by an extension of the bijective mapping (27).

The inversion formulae

$$\lambda = \frac{z_0 - z_1}{z_2 - 1} = \frac{z_2 + 1}{z_0 + z_1},\quad \mu = \frac{z_0 - z_1}{z_2 + 1} = \frac{z_2 - 1}{z_0 + z_1} \tag{28}$$

exhibit the (real or complex) lines $\lambda$ =cst and $\mu$ =cst as the two systems of (real or complex) linear generatrices of $X^{(c)}$.

In the space $\mathbb{C}^2$ of the variables $(\lambda,\mu)$ we introduce the four tubes whose imaginary bases are the coordinate quadrants, denoted as follows in terms of sign-valued functions $\varepsilon_\lambda$ and $\varepsilon_\mu$:

$$\tau^{\varepsilon_\lambda,\varepsilon_\mu} = \{(\lambda,\mu) \in \mathbb{C}^2;\ \varepsilon_\lambda\ \mathrm{Im}\lambda > 0,\ \ \varepsilon_\mu\ \mathrm{Im}\mu > 0,\}.$$

We shall then prove:

**Proposition 7** *Formulae (27) and (28) define biholomorphic mappings from the tubes $\tau^{-,+}$ and $\tau^{+,-}$ onto the respective Lorentz tuboids $\mathcal{T}^+$ and $\mathcal{T}^-$ and from the "pierced" tubes $\tau^{+,+} \setminus \delta$ and $\tau^{-,-} \setminus \delta$ onto the respective chiral tuboids $\mathcal{T}_{\leftarrow}$ and $\mathcal{T}_{\rightarrow}$.*

**Proof.** We first compute the bilinear form $[z \cdot \xi] = z_0\xi_0 - z_1\xi_1 - z_2\xi_2$ for $z \in X^{(c)}$ and $\xi = \xi(\alpha) = (1, \cos\alpha, \sin\alpha) \in C^+$, $|\alpha| \leq \pi$. In view of (27), one obtains, for $|\alpha| \neq \pi$:

$$[z(\lambda,\mu) \cdot \xi(\alpha)] = 2\cos^2\alpha/2\ \frac{(\lambda - \tan\alpha/2)(\mu - \tan\alpha/2)}{\lambda - \mu}, \tag{29}$$

or by putting

$$\lambda_\alpha = -\frac{1}{\lambda - \tan\alpha/2}, \quad \mu_\alpha = -\frac{1}{\mu - \tan\alpha/2} \tag{30}$$

it follows

$$[z\cdot\xi(\alpha)] = 2\cos^2\alpha/2\ \frac{1}{\lambda_\alpha - \mu_\alpha}. \tag{31}$$

a) Since for $|\alpha| \neq \pi$, the quantities $\mathrm{Im}\zeta$ and $\mathrm{Im}\zeta_\alpha$ have the same sign, it follows from Eq.(31) that the condition $(\lambda,\mu) \in \tau^{-,+}$ (resp. $(\lambda,\mu) \in \tau^{+,-}$) implies for $z = z(\lambda,\mu)$ the inequality $\mathrm{Im}([z\cdot\xi(\alpha)]) > 0$ (resp. $\mathrm{Im}([z\cdot\xi(\alpha)]) < 0$ ) for all values of $\alpha$ between $-\pi$ and $\pi$. For $\alpha = \pm\pi$, one has $[z\cdot\xi(\pm\pi)] = z_0 + z_1 = 2\ (\lambda-\mu)^{-1}$ and the same implications still hold. Therefore, in view of the characterization of $\mathcal{T}^\pm$ given in Proposition 1, we have shown that the images of $\tau^{-,+}$ and $\tau^{+,-}$ by the mapping $(\lambda,\mu) \to z(\lambda,\mu)$ are respectively contained in $\mathcal{T}^+$ and $\mathcal{T}^-$.

b) We shall now show that for any point $(\lambda,\mu)$ in $\tau^{+,+} \setminus \delta$, there exist two distinct real numbers $t_+ = \tan\alpha_+/2$ and $t_- = \tan\alpha_-/2$ such that the corresponding complex numbers (defined by (30)) $\lambda_{\alpha_\pm}$, $\mu_{\alpha_\pm}$ satisfy the equalities

$$\mathrm{Im}(\lambda_{\alpha_+} - \mu_{\alpha_+}) = \mathrm{Im}(\lambda_{\alpha_-} - \mu_{\alpha_-}) = 0, \tag{32}$$

so that (in view of (31)) the corresponding image $z = z(\lambda,\mu)$ satisfies the equalities:

$$\mathrm{Im}\left([z\cdot\xi(\alpha_+)]\right) = \mathrm{Im}\left([z\cdot\xi(\alpha_-)]\right) = 0 \tag{33}$$

In fact, being given $\lambda$ and $\mu$ distinct in the upper half-plane, the numbers $t_+$ and $t_-$ are determined by the following geometrical procedure: there are two circles $\gamma_+$ and $\gamma_-$ which contain the points $\lambda$ and $\mu$ and are tangent to the real axis at the respective points $t_+$ and $t_-$. These points lead to the desired condition for the following reason: the inversion $\zeta - t_+ \to -(\zeta - t_+)^{-1}$ transforms the circle $\gamma_+$ into a straight line parallel to the real axis ; therefore the points $\lambda_{\alpha_+}$ and $\mu_{\alpha_+}$ are on this line and thereby satisfy Eq.(32) (the same holds for $t_-$). Note that if $\mathrm{Im}(\lambda-\mu) = 0$, one of the points, say $t_+$, is rejected at infinity: this is the case when $\alpha_+ = \pm\pi$, Eq. (33) being still valid.

Since Eq.(33) holds, it results from the characterization of $\mathcal{T}_\chi$ given in Proposition 4 that the image of $\tau^{+,+} \setminus \delta$ by the mapping $(\lambda,\mu) \to z(\lambda,\mu)$ is contained in $\mathcal{T}_\chi$. It remains to check that it is contained in the chiral component $\mathcal{T}_\leftarrow$ of the latter, defined by (14); in fact, one has: $\epsilon(z) = (i/2)(z_1\overline{z}_2 - z_2\overline{z}_1) = \left[\mathrm{Im}\lambda(1+|\mu|^2) + \mathrm{Im}\mu(1+|\lambda|^2)\right] \times |\lambda-\mu|^{-2}$, which is positive for all $(\lambda,\mu)$ in $\tau^{+,+} \setminus \delta$. Similarly, the image of $\tau^{-,-} \setminus \delta$ is shown to be contained in $\mathcal{T}_\rightarrow$.

Finally, in view of (28), the set of points $(\lambda,\mu)$ in $\mathbb{C}^2$ such that either $\lambda$ or $\mu$ is real represent complex points $z = x = iy$ of $X^{(c)}$ which belong to the complexified linear generatrices of $X$: all these points are such that $y^2 = 0$, namely they belong to the set $\mathcal{T}_0 \setminus \{z \in X^{(c)};\ z_0 + z_1 = 0\}$.

So we have established that the biholomorphic mapping defined by Eqs (27), (28), maps respectively each set of the partition of $\mathbb{C}^2_{(\lambda,\mu)} \setminus \delta$ composed of the tubes $\tau^{-,+}, \tau^{+,-}$, $\tau^{-,-} \setminus \delta$, $\tau^{+,+} \setminus \delta$ and of their interfaces into a corresponding set of the partition of $X^{(c)} \setminus \{z \in X^{(c)};\ z_0 + z_1 = 0\}$ composed of the tuboids $\mathcal{T}^+, \mathcal{T}^-$, $\mathcal{T}_\rightarrow$, $\mathcal{T}_\leftarrow$ and of $\mathcal{T}_0 \setminus \{z \in X^{(c)};\ z_0 + z_1 = 0\}$. Since the whole space $\mathbb{C}^2_{(\lambda,\mu)} \setminus \delta$ is biholomorphically mapped *onto* the whole manifold $X^{(c)} \setminus \{z \in X^{(c)};\ z_0 + z_1 = 0\}$, it follows that the mapping is a *bijection* for all corresponding sets of the previous partitions, which ends the proof of the Proposition. ■

**Remark 1** *The previous proposition shows that, while the tuboids $\mathcal{T}^+$ and $\mathcal{T}^-$ are simply-connected domains, the other two tuboids $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$ admit a nontrivial homotopy generator, corresponding to a loop around the set $\delta$ in their representations by the domains $\tau^{-,-} \setminus \delta$, $\tau^{+,+} \setminus \delta$. On the manifold $X^{(c)}$, a typical representative of such a generator is the "circle" of the plane $z_0 = 0$ parametrized by $z_1 = \sin(u + iv_0)$, $z_2 = \cos(u + iv_0)$; in fact, in view of Proposition 5 (or of the representation (18), (20), (21) with $\Psi = 0$), such a circle is contained in $\mathcal{T}_\rightarrow$ or in $\mathcal{T}_\leftarrow$ according to whether the constant $v_0$ is positive or negative (note that when $z$ describes this circle, the variable $\lambda - \mu = 2/z_1$ describes a loop around the origin).*

### 3.2 Holomorphic functions in the tuboids and integral representations

We shall make use of the following relations which are direct consequences of (4) and (27):

$$d\sigma(z(\lambda,\mu)) = \frac{d(z_0+z_1)dz_2}{2\,|z_0+z_1|} = \frac{d\lambda d\mu}{(\lambda-\mu)^2} \tag{34}$$

and

$$[z(\lambda,\mu) - z(\lambda',\mu')]^2 = -\frac{4(\lambda-\lambda')(\mu-\mu')}{(\lambda-\mu)(\lambda'-\mu')} \tag{35}$$

In view of (34), the Hilbert space $\mathcal{H}(X)$ of functions $f(x)$ on $X$ which are square-integrable with respect to the measure $d\sigma(x)$ can be represented by the space $\hat{\mathcal{H}}$ of functions $\hat{f}(\lambda,\mu) = f(x(\lambda,\mu))$ in $L^2\left(\mathbb{R}^2,\ \frac{d\lambda d\mu}{(\lambda-\mu)^2}\right)$.

For each of the four tuboids $\mathcal{T}^+, \mathcal{T}^-$, $\mathcal{T}_\rightarrow$, and $\mathcal{T}_\leftarrow$ of $X^{(c)}$, we now denote respectively $H^2(\mathcal{T}^+)$, $H^2(\mathcal{T}^-)$, $H^2(\mathcal{T}_\rightarrow)$ and $H^2(\mathcal{T}_\leftarrow)$ the space of functions $F(z)$, which enjoy the following properties:

a) $F$ is holomorphic in the tuboid considered,

b) $F$ admits a boundary value $f(x)$ on $X$ from this tuboid, which belongs to $\mathcal{H}(X)$,

c) $F$ is "sufficiently regular at infinity in its domain" in the following sense: the inverse image $\hat{F}(\lambda,\mu) = F(z(\lambda,\mu))$ of $F$ is such that $(\lambda-\mu)^{-1}\hat{F}(\lambda,\mu)$ belongs to the Hardy space $H^2(\tau^{\epsilon_\lambda,\epsilon_\mu})$ of the corresponding tube $\tau^{\epsilon_\lambda,\epsilon_\mu}$ (with $\epsilon_\lambda, \epsilon_\mu = \pm 1$ given by the prescription of Proposition 7).

Under these assumptions, we can therefore write the following Cauchy integral representation in two variables for the function $(\lambda-\mu)^{-1}\hat{F}(\lambda,\mu)$:

$$\frac{\hat{F}(\lambda,\mu)}{\lambda-\mu} = -\epsilon_\lambda\epsilon_\mu \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{\hat{f}(\lambda',\mu')}{\lambda'-\mu'}\frac{d\lambda' d\mu'}{(\lambda'-\lambda)(\mu'-\mu)}. \tag{36}$$

By rewriting the latter as follows:

$$\hat{F}(\lambda,\mu) = -\epsilon_\lambda\epsilon_\mu \frac{1}{\pi^2}\int_{\mathbb{R}^2} \hat{f}(\lambda',\mu')\frac{(\lambda-\mu)(\lambda'-\mu')}{4(\lambda-\lambda')(\mu-\mu')}\ \frac{d\lambda' d\mu'}{(\lambda'-\mu')^2}, \tag{37}$$

we can then take Eqs (34) and (35) into account and obtain the following Cauchy-type representation for the functions of the previous spaces, holomorphic in either one of the four tuboids $\mathcal{T}^+, \mathcal{T}^-$, $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$ of $X^{(c)}$:

$$F(z) = \frac{\mp 1}{\pi^2}\int_X \frac{f(x)}{(x-z)^2}\ d\sigma(x) \tag{38}$$

In the r.h.s. of the latter, the sign − corresponds to the case of the tuboids $\mathcal{T}^+$ and $\mathcal{T}^-$, while the sign + corresponds to the case of the tuboids $\mathcal{T}_\rightarrow$ and $\mathcal{T}_\leftarrow$.

**Remark 2** *One checks that for every point $x \in X$ the singular set of the Cauchy kernel of (38), namely $\{z;\ (x-z)^2 = 0\}$, does not intersect the previous four tuboids. In fact, this singular set coincides with the intersection of $X^{(c)}$ with its analytic tangent plane at $x$ (whose equation is $[(z-x)\cdot x] = [z\cdot x]+1 = 0$); it is therefore composed of the two linear generatrices of $X$ containing $x$, whose complex points are all in $\mathcal{T}_0$.*

### 3.3 Invariance properties of the spaces $\mathcal{H}(X)$ and $H^2(\mathcal{T}^+)$, $H^2(\mathcal{T}^-)$, $H^2(\mathcal{T}_\rightarrow)$, $H^2(\mathcal{T}_\leftarrow)$

The invariance of the measure $d\sigma(x)$ and of the Cauchy kernel $\left[(x-z)^2\right]^{-1}$ under the group $G$ is equivalently represented by the conformal invariance of the corresponding quantities expressed in terms of the previous set of variables $(\lambda,\mu)$. Apart from the invariance under the translations and dilatations,

which is trivial, we shall stress the invariance under the homographic transformations $(\lambda, \mu) \to (\lambda_\alpha, \mu_\alpha)$ defined by Eq. (30) and whose inverses are given by the formulae

$$\lambda = \tan\alpha/2 - 1/\lambda_\alpha,\ \ \mu = \tan\alpha/2 - 1/\mu_\alpha$$

In fact, one checks that one has for all values of $\alpha$:

$$\mathrm{d}\sigma(x(\lambda,\mu)) = \frac{\mathrm{d}\lambda \mathrm{d}\mu}{(\lambda-\mu)^2} = \frac{\mathrm{d}\lambda_\alpha \mathrm{d}\mu_\alpha}{(\lambda_\alpha-\mu_\alpha)^2} \tag{39}$$

and

$$[z(\lambda,\mu) - z(\lambda',\mu')]^2 = -\frac{4(\lambda-\lambda')(\mu-\mu')}{(\lambda-\mu)(\lambda'-\mu')} = -\frac{4(\lambda_\alpha-\lambda'_\alpha)(\mu_\alpha-\mu'_\alpha)}{(\lambda_\alpha-\mu_\alpha)(\lambda'_\alpha-\mu'_\alpha)} \tag{40}$$

In view of (39), the Hilbert space $\mathcal{H}(X)$ can then be represented, for all $\alpha's$, by the space of functions $\hat{f}_\alpha(\lambda_\alpha, \mu_\alpha) = f(x(\tan\alpha/2 - 1/\lambda_\alpha, \tan\alpha/2 - 1/\mu_\alpha))$ in $L^2\left(\mathbb{R}^2,\ \frac{\mathrm{d}\lambda_\alpha \mathrm{d}\mu_\alpha}{(\lambda_\alpha-\mu_\alpha)^2}\right)$.

Similarly each function $F(z)$ in either one of the spaces $H^2(\mathcal{T}^+)$, $H^2(\mathcal{T}^-)$, $H^2(\mathcal{T}_\to)$, or $H^2(\mathcal{T}_\leftarrow)$ is represented for all values of $\alpha$ by a holomorphic function $\hat{F}_\alpha(\lambda_\alpha, \mu_\alpha) = F(z(\tan\alpha/2 - 1/\lambda_\alpha, \tan\alpha/2 - 1/\mu_\alpha))$ whose domain is the corresponding tube $\tau^{\epsilon_\lambda,\epsilon_\mu}$. Moreover, in view of (39) and (40), the formula (36) is seen to be invariant under the transformation $(\lambda, \mu) \to (\lambda_\alpha, \mu_\alpha)$, which shows that each function $(\lambda_\alpha - \mu_\alpha)^{-1}\hat{F}_\alpha(\lambda_\alpha, \mu_\alpha)$ belongs to the same Hardy space $H^2(\tau^{\epsilon_\lambda,\epsilon_\mu})$.

*This invariance allows us legitimately to call the spaces $H^2(\mathcal{T}^+)$, $H^2(\mathcal{T}^-)$, $H^2(\mathcal{T}_\to)$ and $H^2(\mathcal{T}_\leftarrow)$ Hardy spaces of the corresponding tuboids of $X^{(c)}$.*

## 3.4 Decomposition in Hardy spaces of the four tuboids

Every function $\hat{f}(\lambda, \mu)$ in $\hat{\mathcal{H}}$ admits a decomposition of the form

$$\hat{f} = \hat{f}^{+,+} + \hat{f}^{-,-} + \hat{f}^{+,-} + \hat{f}^{-,+}, \tag{41}$$

where each function $\hat{f}^{\epsilon_\lambda,\epsilon_\mu}$ is the boundary value of a holomorphic function $\hat{F}^{\epsilon_\lambda,\epsilon_\mu}$, such that $(\lambda - \mu)^{-1}\hat{F}(\lambda, \mu)$ belongs to the Hardy space $H^2(\tau^{\epsilon_\lambda,\epsilon_\mu})$ of the corresponding tube $\tau^{\epsilon_\lambda,\epsilon_\mu}$. Each function $\hat{F}^{\epsilon_\lambda,\epsilon_\mu}$ satisfies the Cauchy integral representation (36) and is also directly defined in terms of $\hat{f}$ by the same Cauchy integral in which $\hat{f}$ is substituted to $\hat{f}^{\epsilon_\lambda,\epsilon_\mu}$. This standard result is most simply obtained by considering the four holomorphic functions $(\lambda - \mu)^{-1}\hat{F}^{\epsilon_\lambda,\epsilon_\mu}(\lambda, \mu)$ as the inverse Fourier-Laplace transforms of the Fourier transform of $(\lambda - \mu)^{-1}\hat{f}(\lambda, \mu)$, chopped with the characteristic functions of the four quadrants of Fourier coordinates.

By applying the results of subsections 3.2 and 3.3, and in particular formula (38), we then immediately obtain the following

**Theorem 1** *Every function $f(x)$ in $\mathcal{H}(X)$ admits a decomposition of the form*

$$f = f^+ + f^- + f_\to + f_\leftarrow = \sum\nolimits_{tub} f_{(tub)}, \tag{42}$$

*in which each of the four components $f_{(tub)}(x)$ is the boundary value in $\mathcal{H}(X)$ of a holomorphic function $F_{(tub)}(z)$ belonging to the corresponding Hardy space $H^2(\mathcal{T}^+)$, $H^2(\mathcal{T}^-)$, $H^2(\mathcal{T}_\to)$ and $H^2(\mathcal{T}_\leftarrow)$. Moreover, each function $F_{(tub)}(z)$ is given in its tuboid by the following integral representations (expressed either in terms of $f$ or of its own boundary value $f_{(tub)}$):*

$$F_{(tub)}(z) = \epsilon_{(tub)}\ \frac{1}{\pi^2}\int_X \frac{f(x)}{(x-z)^2}\ \mathrm{d}\sigma(x) = \epsilon_{(tub)}\ \frac{1}{\pi^2}\int_X \frac{f_{(tub)}(x)}{(x-z)^2}\ \mathrm{d}\sigma(x), \tag{43}$$

*in which the sign function $\epsilon_{(tub)}$ takes the value – for the tuboids $\mathcal{T}^+, \mathcal{T}^-$, and + for the tuboids $\mathcal{T}_\to$ and $\mathcal{T}_\leftarrow$.*

We end this section by introducing dense subspaces of the previous spaces which will be of current use in the study of the Fourier-Helgason transformation.

**Definition 1** *We call* $\mathcal{H}_{(reg)}(X)$ *and* $H^2_{(reg)}(\mathrm{tub})$, *where* tub *stands for either one of the four tuboids* $\mathcal{T}^+$, $\mathcal{T}^-$, $\mathcal{T}_\rightarrow$, $\mathcal{T}_\leftarrow$, *the respective subspaces of* $\mathcal{H}(X)$ *and* $H^2(\mathrm{tub})$ *which are represented in the variables* $(\lambda, \mu)$ *by the set of all functions* $\hat{f}(\lambda, \mu)$ *in the corresponding space* $\hat{\mathcal{H}}$ *or* $H^2(\tau^{\epsilon_\lambda, \epsilon_\mu})$ *satisfying the following boundedness property:*
$(\lambda - \mu)^{-1}\hat{f}(\lambda, \mu)$ *(resp.* $(\lambda - \mu)^{-1}\hat{F}(\lambda, \mu)$*) admits a uniform bound on* $\mathbb{R}^2$ *(resp. in the closure of the tube* $\tau^{\epsilon_\lambda, \epsilon_\mu}$*) of the form* $\mathrm{cst} \times (1 + |\lambda|)^{-1}(1 + |\mu|)^{-1}$, *where* cst *denotes an arbitrary constant.*

# 4 The Lorentzian Fourier-Helgason transformation

## 4.1 Definition and properties of the transforms $\tilde{f}_+, \tilde{f}_-$

For all functions $f(x)$ in $\mathcal{H}_{(reg)}(X)$ we introduce the following pair of transforms

$$\tilde{f}_\pm(\xi, s) = \int_X [x_\pm \cdot \xi]^s f(x) \, \mathrm{d}\sigma(x) \tag{44}$$

in which $s$ is a complex parameter with appropriate range, $\xi$ varies on the cone $C^+$ and the kernels $[x_+ \cdot \xi]^s$, $[x_- \cdot \xi]^s$ are defined, for each $\xi \in C^+$, as the boundary values of the holomorphic function $[z \cdot \xi]^s = [(x + iy) \cdot \xi]^s$ when $z$ tends to the reals from the respective tuboids $\mathcal{T}^+$ and $\mathcal{T}^-$ of $X^{(c)}$. In fact, in view of Proposition 1, for each $\xi \in C^+$ the function $[z \cdot \xi]^s$ is holomorphic in the union of $\mathcal{T}^+$ and $\mathcal{T}^-$, since $[z \cdot \xi]$ takes its values correspondingly in the upper and lower half-planes. However, this statement necessitates the following specification: putting $[z \cdot \xi] = t$, the holomorphic function $t^s$ is always considered in its distinguished sheet over $\mathbb{C}\backslash ]-\infty, -1]$, namely, as being positive on $\mathbb{R}^+$. *Denoting by* $t^s_+$ *and* $t^s_-$ *the boundary values on* $\mathbb{R}$ *of the holomorphic function* $t^s$ *respectively from the upper and lower half-planes*, one then has the identity:

$$[x_\pm \cdot \xi]^s = [x \cdot \xi]^s_\pm = Y([x \cdot \xi])|[x \cdot \xi]|^s + e^{\pm i\pi s} Y(-[x \cdot \xi])|[x \cdot \xi]|^s, \tag{45}$$

in which $Y(t)$ is the Heaviside function ($Y(t) = 1$ for $t > 0$ and $= 0$ for $t < 0$). Note that, below, one will only deal with the case $\mathrm{Re}s > -1$ so that the previous equality (45) will always hold in the sense of functions in $L^1$. (A distribution-like treatment would be necessary only in higher dimensions, as considered in [6]).

The transforms $\tilde{f}_\pm$ of $f$ satisfy the homogeneity property $\tilde{f}_\pm(r\xi, s) = r^s \tilde{f}_\pm(\xi, s)$; by taking Eqs (29) and (34) into account for each vector $\xi = \xi(\alpha) = (1, \cos\alpha, \sin\alpha) \in C^+$, Eq. (44) can be rewritten as follows in terms of the parametrization (27) of $X$:

$$\tilde{f}_\pm(\xi(\alpha), s) = 2^s \int_{\mathbb{R}^2} \left[\frac{\hat{f}(\lambda, \mu)}{\lambda - \mu}\right] \frac{[(\cos\alpha/2)\lambda - (\sin\alpha/2)]^s_\mp [(\cos\alpha/2)\mu - (\sin\alpha/2)]^s_\pm}{(\lambda - \mu)^{s+1}_\mp} \mathrm{d}\lambda \mathrm{d}\mu. \tag{46}$$

or equivalently (in view of Eqs (30),(31) and (39)):

$$\tilde{f}_\pm(\xi(\alpha), s) = (2\cos^2\alpha/2)^s \int_{\mathbb{R}^2} \left[\frac{\hat{f}_\alpha(\lambda_\alpha, \mu_\alpha)}{\lambda_\alpha - \mu_\alpha}\right] \frac{\mathrm{d}\lambda_\alpha \mathrm{d}\mu_\alpha}{(\lambda_\alpha - \mu_\alpha)^{s+1}_\mp} \tag{47}$$

This allows us to state the following

**Proposition 8** *a) For every function* $f$ *in* $\mathcal{H}_{(reg)}(X)$, *the corresponding transforms* $\tilde{f}_+(\xi, s)$ *and* $\tilde{f}_-(\xi, s)$ *are well-defined, continuous and homogeneous of degree* $s$ *with respect to* $\xi$ *in* $C^+$, *and holomorphic with respect to* $s$ *in the strip* $-1 < \mathrm{Re}s < 0$.

*b) For every function* $f$ *in the Hardy space* $H^2_{(reg)}(\mathcal{T}^+)$ *(resp.* $H^2_{(reg)}(\mathcal{T}^-)$*), the corresponding transform* $\tilde{f}_+(\xi, s)$ *(resp.* $\tilde{f}_-(\xi, s)$ *) vanishes.*

**Proof.**

a) The following majorization of the r.h.s. of Eq. (46) results from the uniform bound on $\hat{f}$ (postulated according to Definition 1) together with the fact that $|(\lambda_\alpha - \mu_\alpha)_\mp^{-s}| \le \max(e^{\mp\pi \ \mathrm{Im}s},\ 1) \times |\lambda_\alpha - \mu_\alpha|^{-\mathrm{Re}s}$:

$$\begin{aligned} |\tilde{f}_\pm(\xi(\alpha), s)| &\le \mathrm{Cst}\ \max(e^{\mp\pi \ \mathrm{Im}s},\ 1) \times \cdots \\ \cdots \int_{\mathbb{R}^2} & \frac{|(\cos\alpha/2)\lambda - (\sin\alpha/2)|^{\mathrm{Re}s}}{1+|\lambda|} \frac{|(\cos\alpha/2)\mu - (\sin\alpha/2)|^{\mathrm{Re}s}}{1+|\mu|} \frac{\mathrm{d}\lambda \mathrm{d}\mu}{|\lambda-\mu|^{\mathrm{Re}s+1}}. \end{aligned} \tag{48}$$

The uniform convergence and boundedness of the latter integral in all the intervals $-1+\eta \le \mathrm{Re}s \le -\eta$ (with $\eta > 0$ ) gives the result.

b) Assuming that $f$ belongs to $H^2_{(reg)}(\mathcal{T}^+)$, namely that $\hat{f}(\lambda,\mu)$ is the boundary value of a holomorphic function $\hat{F}$ bounded by cst $\times (1+|\lambda|)^{-1}(1+|\mu|)^{-1}$ in $\tau^{-,+}$, we can distort the integration cycle of (46) into $\mathbb{R}^2 + i(-a,a)$, $a$ being any positive number, *provided one has chosen the prescription $\lambda_-, \mu_+$ (of $\tilde{f}_+$) in the integrand of (46).* The corresponding integral being then independent of $a$, we see by taking the limit $a \to \infty$ that this integral has to be equal to zero for all values of $s$ in the strip $-1 < \mathrm{Re}s < 0$ where it is defined. ■

**Definition 2** *Given a function $f$ in $\mathcal{H}_{(reg)}(X)$, we define its* Lorentzian Fourier-Helgason (FH) transforms *as the restrictions of $\tilde{f}_\pm$ to the symmetry axis $s = -1/2 - i\nu$ of their analyticity domain in $s$, namely the following pair of functions on the cone $C^+$, homogeneous of degree $-1/2 - i\nu$:*

$$\tilde{f}_{\pm,\nu}(\xi) = \tilde{f}_\pm(\xi, -\frac{1}{2} - i\nu) = \int_X [x_\pm \cdot \xi]^{-\frac{1}{2}-i\nu} f(x)\ \mathrm{d}\sigma(x) \tag{49}$$

*In view of proposition 8b), for every function $f \in H^2_{(reg)}(\mathcal{T}^+)$ (resp. $H^2_{(reg)}(\mathcal{T}^-)$), there is a* unique *Lorentzian FH-transform which is $\tilde{f}_{-,\nu}(\xi)$ (resp. $\tilde{f}_{+,\nu}(\xi)$).*

Note that in view of (45), the r.h.s. of Eq. (49) can be read more explicitly as follows:

$$\int_{\{x\in X;[x\cdot\xi]>0\}} |[x\cdot\xi]|^{-\frac{1}{2}-i\nu} f(x)\ \mathrm{d}\sigma(x) \mp i \int_{\{x\in X;[x\cdot\xi]<0\}} e^{\pm\pi\nu} |[x\cdot\xi]|^{-\frac{1}{2}-i\nu} f(x)\ \mathrm{d}\sigma(x). \tag{50}$$

and that one has, in view of (48), the following

**Proposition 9** *The Lorentzian FH-transforms $\tilde{f}_{\pm,\nu}$ of any function $f$ belonging to $\mathcal{H}_{(reg)}(X)$ satisfy uniform bounds of the form:*

$$|\tilde{f}_{\pm,\nu}(\xi)| \quad \le \mathrm{Cst}\ \xi_0^{-1/2}\ \max(e^{\pm\pi\nu},\ 1) \tag{51}$$

## 4.2 Inversion of the transformation

Let $[i_\Xi\omega](\xi)$ be the one-form on $C^+$ obtained by contracting the vector field $\Xi(\xi) = (\xi_0, \xi_1, \xi_2)$ with the $G-$invariant volume form $\omega(\xi) = \frac{\mathrm{d}\xi_0\wedge\mathrm{d}\xi_1\wedge\mathrm{d}\xi_2}{\mathrm{d}(\xi^2)}\Big|_{C^+}$ and let $\mathrm{d}\mu_\gamma$ be the measure obtained by restricting this one-form to any given loop $\gamma$ on $C^+$ homotopic to the circle $\gamma_0 = \{\xi \in C^+; \xi = \xi(\alpha) = (1, \cos\alpha, \sin\alpha),\ -\pi \le 0 \le \pi\}$. One checks in particular that $\mathrm{d}\mu_{\gamma_0} = \mathrm{d}\alpha/2$. With Euler's identity, one checks the following property, of current use below:

**Proposition 10** *For every function $a(\xi)$ on $C^+$ homogeneous of degree $-1$, the one-form $a(\xi)[i_\Xi\omega](\xi)$ is closed.*

We now wish to show:

**Theorem 2** *Let $f(x)$ belong to $H^2_{(reg)}(\mathcal{T}^-)$ (resp. $H^2_{(reg)}(\mathcal{T}^+)$), and let $\tilde{f}_{+,\nu}$ (resp. $\tilde{f}_{-,\nu}$) be its FH-transform. Then the holomorphic function $F(z)$ in $\mathcal{T}^-$ (resp. $\mathcal{T}^+$) whose boundary value is $f$ is given in terms of $\tilde{f}_{+,\nu}$ (resp. $\tilde{f}_{-,\nu}$) by the following formula:*

$$F(z) = \frac{1}{2\pi^2}\int_0^\infty \frac{\nu\tanh\pi\nu}{e^{\pm\pi\nu}\cosh\pi\nu}\mathrm{d}\nu \int_\gamma [z\cdot\xi]^{-\frac{1}{2}+i\nu}\tilde{f}_{\pm,\nu}(\xi)\ \mathrm{d}\mu_\gamma(\xi) \tag{52}$$

One first checks that the double integral at the r.h.s. of formula (52) converges for all $z$ in $\mathcal{T}^-$ (resp. $\mathcal{T}^+$): in fact, in view of Proposition 1, the condition $z\in\mathcal{T}^-$ (resp. $\mathcal{T}^+$) implies that $\left|[z\cdot\xi]^{i\nu}\right| \le e^{-a\nu}$ for some $a = a(z)$ such that $-\pi < a < 0$ (resp. $0 < a < \pi$). This implies, in view of Proposition 9, that the integrand at the r.h.s. of (52) is uniformly bounded by cst $e^{-(\pi+a)\nu}$ (resp. cst $e^{-a\nu}$). The expression at the r.h.s. of Eq. (52) is therefore a holomorphic function of $z$ in $\mathcal{T}^-$ (resp. $\mathcal{T}^+$). Moreover, due to the homogeneity of degree $-\frac{1}{2}-i\nu$ in $\xi$ of $\tilde{f}_{\pm,\nu}$, this holomorphic function is independent of the choice of the cycle $\gamma$, as a consequence of Proposition 10 and Stokes theorem,

In order to prove Theorem 2, we shall make use of the Cauchy-type representation (38) established above together with the following expression of the Cauchy kernel on $X^{(c)}$ which will be proved in our next subsection:

**Proposition 11** *The Cauchy kernel on $X^{(c)}$ is given by the following double integral:*

$$\frac{1}{(z'-z)^2} = -\frac{1}{2}\int_0^\infty \frac{\nu\tanh\pi\nu}{e^{\pm\pi\nu}\cosh\pi\nu}\mathrm{d}\nu\int_\gamma [z\cdot\xi]^{-\frac{1}{2}+i\nu}[\xi\cdot z']^{-\frac{1}{2}-i\nu}\mathrm{d}\mu_\gamma(\xi) \tag{53}$$

*which is absolutely convergent for $(z,z')$ in $\mathcal{T}^-\times\mathcal{T}^+$ and in $\mathcal{T}^+\times\mathcal{T}^-$. This formula remains meaningful when one of the points, e.g. $z'$, is taken real, provided the appropriate limit is taken, namely respectively $z' = x_+$ or $z' = x_-$, in the r.h.s. of (53).*

If we plug the expression (53) of $\left[(x-z)^2\right]^{-1}$ into the r.h.s. of (38), invert the integrals over $x$ and over $(\nu,\xi)$ in this absolutely convergent integral and take into account the defining formula (49) of $\tilde{f}_{\pm,\nu}(\xi)$, we readily obtain formula (52) for both types of configurations $z\in\mathcal{T}^-, z' = x_+$ and $z\in\mathcal{T}^+, z' = x_-$ (corresponding to the cases $f\in H^2_{(reg)}(\mathcal{T}^-)$ and $f\in H^2_{(reg)}(\mathcal{T}^+)$). So, proving Theorem 2 amounts to proving Proposition 11.

## 4.3 Lorentzian invariant perikernels: a new representation for the first-kind Legendre functions and the Cauchy kernel on $X^{(c)}$

An important class of holomorphic kernels on the ($d-$dimensional) complexified hyperboloid has been introduced in ([11]) and ([12]) under the name of "perikernels". [2] Since it plays a basic role in our approach to the FH-transformation, we now recall this notion in the two-dimensional case presently considered.

A perikernel is a holomorphic function $\mathrm{W}(z,z')$ defined in the "cut-domain" $\Delta = X^{(c)}\times X^{(c)}\setminus\Sigma^{(c)}$, where "the cut" $\Sigma^{(c)}$ is the set $\{(z,z')\in X^{(c)}\times X^{(c)} : [z\cdot z']\in[-\infty,-1]\}$. Such a perikernel is *invariant* if it moreover satisfies in $\Delta$ the following condition:

$$\mathrm{W}(gz,gz') = \mathrm{W}(z,z'), \tag{54}$$

for all $g\in G^{(c)}$.

Since $\Delta = \{(z,z')\in X^{(c)}\times X^{(c)} : [z\cdot z']\in\hat{\Theta}_L\}$, it follows from Proposition 3 that W is holomorphic in particular in the two tuboids $\mathcal{T}^{-+} = \mathcal{T}^-\times\mathcal{T}^+$ and $\mathcal{T}^{+-} = \mathcal{T}^+\times\mathcal{T}^-$ of $X^{(c)}\times X^{(c)}$. The corresponding

[2] These kernels arise in the context of quantum field theory on a $d-$dimensional one-sheeted hyperboloid ($d\ge 2$), interpreted as a $d-$dimensional de Sitter spacetime manifold. In [6] we have given a complete study of these kernels $\mathcal{W}(x_1,x_2)$ and characterized them as the "two-point functions" of Wightman quantum field theories on the corresponding de Sitter spacetime.

restrictions $W^{-+}$ and $W^{+-}$ of W admit boundary-values on $X$, denoted respectively $\mathcal{W}^{-+} = \mathcal{W}(x, x')$ and $\mathcal{W}^{+-} = \mathcal{W}(x', x)$ (the symmetry of these two boundary values being a consequence of (54)). These two distributions are such that the difference $\mathcal{C} = \mathcal{W}^{-+} - \mathcal{W}^{+-}$ (i.e. the discontinuity of W across the cut $\Sigma^{(c)}$) vanishes for $[x \cdot x'] > -1$.

In view of its $G-$invariance property (54), $W(z, z')$ can also be identified with a function $w([z \cdot z'])$ holomorphic in the cut-plane $\hat{\Theta}_L$. which can be called a reduced form of W.

Basic examples of invariant perikernels [3] are provided by the first-kind Legendre functions, as shown by the following statement (see also [6] where a generalization to the $d-$dimensional case is given)

**Proposition 12** *The following integral representation holds:*

$$P_{-\frac{1}{2}+i\nu}([z \cdot z']) = \frac{e^{\mp\pi\nu}}{\pi} \int_\gamma [z \cdot \xi]^{-\frac{1}{2}+i\nu} [\xi \cdot z']^{-\frac{1}{2}-i\nu} \mathrm{d}\mu_\gamma(\xi). \tag{55}$$

*In the latter, the class of integration cycles $\gamma$ and the corresponding measures $\mathrm{d}\mu_\gamma$ are those defined in subsection 4.2 and the integral defines a pair of holomorphic functions in the respective domains $\mathcal{T}^{-+}$ and $\mathcal{T}^{+-}$, corresponding to the choice of sign – or + in the exponential in front of the integral. Moreover, formula (55) defines the first-kind Legendre functions as a family of invariant perikernels $W_{(\nu)}(z, z') = w_{(\nu)}([z \cdot z']) = P_{-\frac{1}{2}+i\nu}([z \cdot z'])$ depending on the parameter $\nu$.*

**Proof.** Call $W^{-+}_{(\nu)}(z, z')$ and $W^{+-}_{(\nu)}(z, z')$ the pair of functions defined by the r.h.s. of Eq. (55) as holomorphic functions (in view of Proposition 1) in the respective tuboids $\mathcal{T}^{-+}$ and $\mathcal{T}^{+-}$,

The independence of these functions with respect to the choice of $\gamma$, which is a consequence of proposition 10 and of Stokes theorem, can be used to show that they enjoy the $G-$invariance property. In fact, starting from e.g. $\gamma = \gamma_0$, one sees that all the cycles $\gamma = g\gamma_0$, with $g \in G$, are homotopic to $\gamma_0$, and that each measure $\mathrm{d}\mu_\gamma$ is the transform of $\mathrm{d}\mu_{\gamma_0}$ by the action of $g$ and is itself invariant under the (rotation) subgroup of $G$ which leaves $\gamma$ invariant. This allows one to make the change of variable $\xi \to g\xi$ for any $g \in G$ in (55) and therefore to check that $W^{\mp\pm}_{(\nu)}(z, z') = W^{\mp\pm}_{(\nu)}(gz, gz')$ for all $g \in G$.

In order to compute $W^{\mp\pm}_{(\nu)}(z, z')$ explicitly, we choose $\gamma = \gamma_0$ and a corresponding appropriate configuration in $\mathcal{T}^{-+}$, namely the pair $(z = (-i\cosh v, 0, i\sinh v),\ \ z' = (i, 0, 0))$, such that $[z \cdot z'] = \cosh v \in \mathbb{R}_+$. We then obtain:

$$w_{(\nu)}(\cosh v) = \frac{1}{\pi} \int_{-\pi}^{\pi} (\cosh v + \sinh v \sin\alpha)^{-\frac{1}{2}+i\nu} \frac{\mathrm{d}\alpha}{2} = P_{-\frac{1}{2}+i\nu}(\cosh v). \tag{56}$$

The latter equality results from a standard integral representation of the Legendre function of first kind [18]. Of course, by taking the complex conjugate configurations, one would justify similarly the corresponding form of Eq. (55) valid for $(z, z') \in \mathcal{T}^{+-}$. ■

**Remark 3** *The function $W_{(\nu)}(z, z')$ is a solution in both variables of the Laplace-Beltrami equation $[\Delta_X - (\nu^2 + \frac{1}{4})]W_{(\nu)}(z, z') = 0$ on the manifold $X^{(c)}$ (or in other words, the Klein-Gordon equation on complexified de Sitter spacetime). For such a $G-$invariant solution, this equation reduces to a Legendre equation in the variable $[z \cdot z'] = \cosh v$ and the required analyticity domain $\hat{\Theta}_L = \mathbb{C}\backslash ]-\infty, -1]$ selects the first-kind Legendre function $P_{-\frac{1}{2}+i\nu}(\cosh v)$ as the unique relevant solution (up to a constant factor).*

We now come back to the Cauchy kernel $[(z - z')^2]^{-1} = \{-2([z \cdot z'] + 1)\}^{-1}$, whose analyticity in $\hat{\Theta}_L$ and $G-$invariance show that it defines an invariant perikernel on $X^{(c)}$: we prove that the r.h.s. of Eq. (53) is precisely a representation of this perikernel $\mathcal{K}(z, z')$ in the pair of tuboids $\mathcal{T}^{-+}$ and $\mathcal{T}^{+-}$.

**Proof of Proposition 11**

[3] In the application to de Sitter quantum field theory, these examples are interpreted as the two-point functions (or "propagators") of massive Klein-Gordon-like fields [6] up to an appropriate normalization.

**A)** We first give a fast proof which is based on the integral representation (55) of the first-kind Legendre functions and on the following Mehler's formula ([18], Eq. 3.14.7, taken for $y = 1, p \to -\nu$):

$$\frac{1}{1-x} = \pi \int_0^\infty \frac{\nu \tanh \pi\nu}{\cosh \pi\nu} \; P_{-\frac{1}{2}-i\nu}(-x) \; \mathrm{d}\nu, \tag{57}$$

which yields for $[z \cdot z'] = -x \in \hat{\Theta}_L$ (since $z^2 = z'^2 = -1$):

$$\frac{1}{(z'-z)^2} = -\frac{\pi}{2} \int_0^\infty \frac{\nu \tanh \pi\nu}{\cosh \pi\nu} \; P_{-\frac{1}{2}-i\nu}([z \cdot z']) \; \mathrm{d}\nu. \tag{58}$$

By plugging the expression (55) of $P_{-\frac{1}{2}-i\nu}([z \cdot z'])$ into the r.h.s. of (58), one readily obtains (53) with the specifications of Proposition 11.

**Remark 4** *It follows from the well-known symmetry property of Legendre functions* $P_{-\frac{1}{2}-i\nu} = P_{-\frac{1}{2}+i\nu}$, *that the integration range over* $\nu$ *can be as well replaced by* $[-\infty, +\infty]$ *(together with an extra factor* $1/2$*) in formula (53). We shall recover this result and see another aspect of it below.*

**B)** We now give an alternative direct proof which exhibits other aspects of the Cauchy kernel on $X$.

Consider the kernel $\mathcal{K}(z, z')$ defined by the r.h.s. of Eq. (53) for $z \in \mathcal{T}^-$ and $z' \in \mathcal{T}^+$. The same argument as in the proof of Proposition 12 shows the $G-$invariance of this kernel and also allows the use of special configurations, such as ($z = (-i\cosh v, 0, i \sinh v)$, $z' = (i, 0, 0)$) and the integration cycle $\gamma_0$, for making the computations simpler; we shall exploit this possibility below.

At first, we invert the integrals over $\nu$ and over $\xi$ and thus reexpress $\mathcal{K}$ as follows:

$$\mathcal{K}(z, z') = -\frac{1}{2} \int_\gamma \mathrm{d}\mu_\gamma(\xi) \int_0^{+\infty} [z \cdot \xi]^{-\frac{1}{2}+i\nu} [\xi \cdot z']^{-\frac{1}{2}-i\nu} \nu e^{-\pi\nu} \frac{\tanh\, \pi\nu}{\cosh\, \pi\nu} \; \mathrm{d}\nu \tag{59}$$

Let us put: $[z \cdot \xi] = A(\xi, z) \quad [z' \cdot \xi] = B(\xi, z') \quad a = \log\, A, \quad b = \log\, B$ and

$$w[\xi, z, z'] = (a - b + i\pi) = \log \left( \frac{-[z \cdot \xi]}{[z' \cdot \xi]} \right). \tag{60}$$

Note that for $z \in \mathcal{T}^-$ and $z' \in \mathcal{T}^+$, one has $0 < \mathrm{Arg}\; B < \pi$ and $-\pi < \mathrm{Arg}\; A < 0$, and therefore (since $\mathrm{Im}\; w = \pi + \mathrm{Arg}\; A - \mathrm{Arg}\; B$): $|\mathrm{Im}\; w| < \pi$. This implies the convergence and uniform boundedness of the integral

$$J_+(w) = \int_0^{+\infty} e^{iw\nu} \;\; \frac{\nu\, \tanh\, \pi\nu}{\cosh\, \pi\nu} \mathrm{d}\nu \tag{61}$$

which allows one to rewrite the expression (59) of $\mathcal{K}$ as follows:

$$\mathcal{K}(z, z') = -\frac{1}{2} \int_\gamma \mathrm{d}\mu_\gamma(\xi) \;\; \frac{J_+(w[\xi, z, z'])}{([z \cdot \xi][z' \cdot \xi])^{1/2}}. \tag{62}$$

Choosing the previous special configurations for $z, z'$ (with $[z \cdot z'] = \cosh v$) and $\gamma = \gamma_0$, $w$ becomes a function $w(\alpha, v)$ of the parameter $\alpha$ of $\gamma_0$, such that $[z \cdot \xi][z' \cdot \xi] = \cosh v + \sin\alpha \sinh v = e^w$; this yields the following equivalent forms for the integral (62):

$$\mathcal{K}(z, z') = -\frac{1}{4} \int_{-\pi}^{\pi} \frac{J_+(w(\alpha, v))\; \mathrm{d}\alpha}{(\cosh v + \sin\alpha\; \sinh v)^{1/2}} = -\frac{1}{2} \int_{-v}^{v} \frac{J_+(w)\; \mathrm{d}w}{[2(\cosh v - \cosh w)]^{1/2}}. \tag{63}$$

or equivalently:

$$\mathcal{K}(z, z') = -\frac{1}{4} \int_{-v}^{v} \frac{J(w)\; \mathrm{d}w}{[2(\cosh\; v - \cosh\; w)]^{1/2}}, \tag{64}$$

where

$$J(w) = J_+(w) + J_+(-w) = \int_{-\infty}^{+\infty} e^{iw\nu} \frac{\nu \tanh \pi\nu}{\cosh \pi\nu} \mathrm{d}\nu \tag{65}$$

Integration by parts allows one to rewrite Eq. (65) as follows:

$$J(w) = \frac{\mathrm{d}}{\mathrm{d}w}\left[\frac{w}{\pi}\int_{-\infty}^{\infty} e^{iw\nu}\frac{1}{\cosh \pi\nu}\mathrm{d}\nu\right] = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}w}\left[\frac{w}{\cosh \frac{w}{2}}\right] \tag{66}$$

The latter expression has been obtained by applying the residue method to the intermediate integral (with the cycle $\Gamma$ with support $\mathbb{R} \cup \{\mathbb{R}+i\}$), namely:

$$\int_{-\infty}^{\infty} e^{iw\nu}\frac{1}{\cosh \pi\nu}\mathrm{d}\nu = \frac{1}{1+e^{-w}}\int_{\Gamma} e^{iw\nu}\frac{1}{\cosh \pi\nu}\mathrm{d}\nu = \frac{2e^{-w/2}}{1+e^{-w}} = \frac{1}{\cosh w/2}. \tag{67}$$

We can therefore rewrite the expression (64) of $\mathcal{K}$ as follows:

$$\mathcal{K}(z,z') = -\frac{1}{4\pi}\int_{-v}^{v}\frac{\mathrm{d}w}{[2(\cosh v - \cosh w)]^{1/2}}\frac{\mathrm{d}}{\mathrm{d}w}\left[\frac{w}{\cosh \frac{w}{2}}\right] \tag{68}$$

We now compute this integral by rewriting it equivalently:

$$\mathcal{K}(z,z') = -\frac{1}{8\pi}\int_{-v}^{v}\frac{\mathrm{d}w}{\left[\cosh^2\frac{v}{2} - \cosh^2\frac{w}{2}\right]^{1/2}}\left[\frac{1}{\cosh \frac{w}{2}} - \frac{w}{2}\frac{\tanh \frac{w}{2}}{\cosh \frac{w}{2}}\right] \tag{69}$$

Taking the variable $\tanh \frac{w}{2} = t$ and putting $\tanh \frac{v}{2} = s$ ($|s| \leq 1$) then yields:

$$\mathcal{K}(z,z') = -\frac{1}{4\pi}\left(1-s^2\right)^{1/2}\int_{-s}^{+s}\frac{\mathrm{d}t}{\left(s^2-t^2\right)^{1/2}}\left[1 - \frac{t}{2}\log\frac{1+t}{1-t}\right] \tag{70}$$

or

$$\mathcal{K}(z,z') = -\frac{1}{4\pi}\left(1-s^2\right)^{1/2}[\pi - I(s)] \tag{71}$$

with

$$I(s) = \int_{-s}^{s}\frac{\mathrm{d}t}{\left(s^2-t^2\right)^{1/2}}\ t\log(1+t)\mathrm{d}t. \tag{72}$$

Integration by parts yields:

$$I(s) = \int_{-s}^{s}\frac{\left(s^2-t^2\right)^{1/2}}{1+t}\mathrm{d}t, \tag{73}$$

$$\frac{\mathrm{d}I(s)}{\mathrm{d}s} = s\int_{-s}^{s}\frac{\mathrm{d}t}{(1+t)\left(s^2-t^2\right)^{1/2}} = \frac{s}{2}\oint\frac{\mathrm{d}t}{(1+t)\left(s^2-t^2\right)^{1/2}} \tag{74}$$

In the latter integral, the integration is done on a clockwise contour surrounding the interval $[-s,s]$ and the residue theorem yields:

$$\frac{\mathrm{d}I(s)}{\mathrm{d}s} = \frac{\pi s}{\left(1-s^2\right)^{1/2}}. \tag{75}$$

Therefore (since $I(0)=0$) we obtain $I(s) = -\pi\left(1-s^2\right)^{1/2} + \pi$ and in view of (71):

$$\mathcal{K}(z,z') = -\frac{1}{4}\left(1-s^2\right) = -\frac{1}{4\cosh^2\frac{v}{2}} = \frac{1}{(z-z')^2} \tag{76}$$

(A similar computation holds for complex conjugate configurations $(z,z') \in \mathcal{T}^{+-}$).

*A geometrical interpretation of the integral (68):*

By taking into account the earlier definition (60) of the variable $w$ as $w[\xi, z, z']$, we see that the integral over $w$ in (68) can be transformed back to an integral over $\xi$ similar to (59) or (62) on a general cycle $\gamma$, but in which the integral over $\nu$ is equal to $J(w)$ instead of $J_+(w)$ (corresponding to the integration range $\nu \in [-\infty, +\infty]$). This yields:

$$\mathcal{K}(z, z') = -\frac{1}{4\pi} \int_\gamma \frac{\mathrm{d}\mu_\gamma(\xi)}{([z \cdot \xi][z' \cdot \xi])^{1/2}} \frac{\mathrm{d}}{\mathrm{d}w} \left[ \frac{w}{\cosh \frac{w}{2}} \right] (\xi, z, z') \tag{77}$$

and justifies the following alternative form to Proposition 11

**Proposition 13** *The Cauchy kernel on $X^{(c)}$ admits the following alternative expressions*

$$\frac{1}{(z'-z)^2} = -\frac{1}{4} \int_{-\infty}^{\infty} \frac{\nu \tanh \pi\nu}{e^{\pm\pi\nu} \cosh \pi\nu} \mathrm{d}\nu \int_\gamma [z \cdot \xi]^{-\frac{1}{2}+i\nu} [\xi \cdot z']^{-\frac{1}{2}-i\nu} \mathrm{d}\mu_\gamma(\xi) \tag{78}$$

$$= -\frac{1}{4\pi i} \int_\gamma \mathrm{d}\mu_\gamma(\xi) \left[ \frac{1}{[(z-z') \cdot \xi]} - \frac{\log\left(\frac{-[z\cdot\xi]}{[z'\cdot\xi]}\right) \ [(z+z') \cdot \xi]}{[(z-z') \cdot \xi]^2} \right], \tag{79}$$

*valid for all $(z, z')$ in $\mathcal{T}^{-+}$ (resp. in $\mathcal{T}^{+-}$).*

We notice that the latter expression (79) of the Cauchy kernel (obtained from Eq. (77) by replacing $w$ by its expression (60)) has the form of a Cauchy- Fantappié-like kernel, interpretable as an integral over a set of analytic singularities (namely the hyperplanes with equation $[(z - z') \cdot \xi] = 0$) which are exterior, but tangential, to the holomorphy domains $\mathcal{T}^{-+}$ and $\mathcal{T}^{+-}$ of $X^{(c)}$.

**Remark 5** *The inversion formula (52) given in Theorem 2 can equivalently be replaced by the following one*

$$F(z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{\nu \tanh \pi\nu}{e^{\pm\pi\nu} \cosh \pi\nu} \mathrm{d}\nu \int_\gamma [z \cdot \xi]^{-\frac{1}{2}+i\nu} \tilde{f}_{\pm,\nu}(\xi) \mathrm{d}\mu_\gamma(\xi) \tag{80}$$

*although the equality of the subintegrals over $\nu$ on $[0, +\infty[$ and $]-\infty, 0]$ is not a trivial symmetry of the formula, but results from the symmetry $\nu \to -\nu$ which appeared in the computation of the Cauchy kernel (see Remark 4). In fact, this shows that the knowledge of $F(z)$ (for e.g. $z \in \mathcal{T}^-$) is entirely encoded in the knowledge of $\tilde{f}_{+,\nu}$ for all positive $\nu$, or for all negative $\nu$, although $\tilde{f}_{+,\nu}$ and $\tilde{f}_{+,-\nu}$ are not related in a simple way.*

# 5 Fourier-Helgason transform of $G_b-$invariant functions and spherical Laplace transform of Volterra kernels

Our aim is to apply the previous definition (49) of the Fourier-Helgason transformation to a particular class of boundary values of holomorphic functions in the tuboids $\mathcal{T}^+$ and $\mathcal{T}^-$: these functions, denoted $\underline{\mathrm{W}}^{\pm}(z)$, enjoy the additional property of being invariant under the stabilizer subgroup $G_b^{(c)}$ of the base point $b = (0, 0, 1)$.

Let us recall (see [11] and [12]) how such functions occur as representatives of invariant perikernels and at first what is the relationship of the latter with the invariant Volterra kernels on $X$ introduced in [13], [14].

1) Let $\mathrm{W}(z, z') = w([z \cdot z'])$ be any invariant perikernel, holomorphic in the domain $\Delta$ of $X^{(c)} \times X^{(c)}$ (see subsection 4.3), $\mathcal{W}^{-+}$ and $\mathcal{W}^{+-}$ its boundary values on $X \times X$ from the respective tuboids $\mathcal{T}^{-+}$ and $\mathcal{T}^{+-}$. Their difference $\mathcal{C} = \mathcal{W}^{-+} - \mathcal{W}^{+-}$ has its support contained in the set $\{(x, x') \in X \times X;\ (x - x')^2 \geq 0\}$. This set admits a splitting into two $G-$invariant closed parts, having only in common the diagonal set $\{(x, x'); x = x'\}$ and characterized by the respective conditions $x \geq x'$ and $x' \geq x$ ($\geq$ being the order

relation on $X$ defined at the beginning of section 2). One can thus write a decomposition of $\mathcal{C}$ of the form $\mathcal{C} = -i\,(\mathcal{R} - \mathcal{A})$, such that the support of $\mathcal{R}$ (resp. of $\mathcal{A}$) is contained in the set $\{(x, x') \in X \times X;\ x \geq x'\}$ (resp. $\{(x, x') \in X \times X;\ x' \geq x\}$). *$\mathcal{R}$ (and also $\mathcal{A}$) is then a Volterra kernel on $X$ in the sense of [14], which is associated with the perikernel* W. In view of the time-ordering interpretation of the relation $x \geq x'$, $\mathcal{R}$ and $\mathcal{A}$ are called respectively "retarded " and "advanced" kernels.

2) By using the transitivity of the group $G^{(c)}$ on $X^{(c)}$, one can then identify $\mathrm{W}(z, z')$ with a $G_b^{(c)}$-invariant function $\underline{\mathrm{W}}(z) = \mathrm{W}(z, b)$; $\underline{\mathrm{W}}$ is holomorphic in the domain

$$\underline{\Delta} = \left\{ z \in X^{(c)},\ z_2 = -[z \cdot b] \in \mathbb{C} \setminus [1, \infty[ \right\}. \tag{81}$$

In view of its $G_b^{(c)}-$invariance property, $\underline{\mathrm{W}}(z)$ can also be identified with the function $\mathrm{w}(-z_2)$ holomorphic in the cut-plane $\hat{\Theta}_L = \mathbb{C} \backslash ]-\infty, -1]$. $\underline{\mathrm{W}}$, like w, is a reduced form of W. It follows from Proposition 3 (since $\hat{\Theta}_L \supset \Theta_L$) that $\underline{\mathrm{W}}$ admits restrictions to the tuboids $\mathcal{T}^-$ and $\mathcal{T}^+$, which we call respectively $\underline{\mathrm{W}}^-$ and $\underline{\mathrm{W}}^+$. The corresponding boundary values on $X$, denoted $\underline{\mathcal{W}}^\pm$, are such that $\underline{\mathcal{W}}^-(x) = \mathcal{W}^{-+}(x, b)$ and $\underline{\mathcal{W}}^+(x) = \mathcal{W}^{+-}(x, b)$. The corresponding jump of $\underline{\mathrm{W}}$, namely the function $\underline{\mathcal{C}} = \underline{\mathcal{W}}^- - \underline{\mathcal{W}}^+$ (such that $\underline{\mathcal{C}}(x) = \mathcal{C}(x, b)$), has its support contained in $\Gamma^+(b) \cup \Gamma^-(b) = \{x \in X;\ x_2 \geq 1\}$. The boundary values $\underline{\mathcal{W}}^+$ and $\underline{\mathcal{W}}^-$ therefore coincide in the open set $\mathcal{U}_b = \{x \in X;\ x_2 < 1\}$, being in fact identical with the restriction of the analytic function $\underline{\mathrm{W}}$ to $\mathcal{U}_b$. Similarly, *the associated invariant Volterra kernel $\mathcal{R}$ (resp. $\mathcal{A}$) admits a reduced form* $\underline{\mathcal{R}}(x) = \mathcal{R}(x, b) = r(x_2)Y(x_0)$ (resp. $\underline{\mathcal{A}}(x) = \mathcal{A}(x, b) = a(x_2)Y(-x_0)$). The latter correspond to a decomposition $\underline{\mathcal{C}} = -i\,(\underline{\mathcal{R}} - \underline{\mathcal{A}})$ of $\underline{\mathcal{C}}$, such that the supports of $\underline{\mathcal{R}}$ and $\underline{\mathcal{A}}$ are respectively contained in the sets $\Gamma^+(b)$ (i.e. $x_2 \geq 1,\ x_0 \geq 0$) and $\Gamma^-(b)$ (i.e. $x_2 \geq 1,\ x_0 \leq 0$) (these are the "future cone" and "past cone" of $b$ in $X$). The corresponding function $r(x_2)$ has therefore its support contained in $[1, +\infty[$; $\underline{\mathcal{R}}$ and $r$ (resp. $\underline{\mathcal{A}}$ and $a$) are also called the retarded (resp. advanced) functions associated with $\underline{\mathrm{W}}$; moreover, *the $G_b-$invariance of $\underline{W}$ implies the equality $a(x_2) = r(x_2)$.*

## 5.1 The spherical Laplace transformation as a special case of the Fourier-Helgason transformation.

We shall make use of the following

**Proposition 14** *If a function $f$ on $X$ is invariant under the subgroup $G_b$ of $G$, then its Lorentzian FH-transforms are of the following form:*

$$\tilde{f}_{\pm,\nu}(\xi) = |[\xi \cdot b]|^{-\frac{1}{2} - i\nu} \left( Y([\xi \cdot b])\tilde{F}_\pm(\nu) + Y(-[\xi \cdot b])\tilde{F}'_\pm(\nu) \right) \tag{82}$$

**Proof.** It is clear from the defining formula (49) that if $f(x)$ is invariant under $G_b$, then $\tilde{f}_{\pm,\nu}(\xi)$ is also invariant under $G_b$, considered as acting on the cone $C^+$. Now there are two classes of orbits of $G_b$ on $C^+$, which are generated respectively by the action of the positive dilatations $\xi \to r\xi,\ r > 0$, on the following special orbits: $\omega = \{\xi = (\cosh \chi, \sinh \chi, -1)\}$ and $\omega' = \{\xi = (\cosh \chi, \sinh \chi, 1)\}$. Then (provided they are defined) the Lorentzian FH-transforms of $f$ are constant on these orbits $\omega$ and $\omega'$, namely one can put $\tilde{f}_{\pm,\nu}\Big|_\omega = \tilde{F}_\pm(\nu)$ and $\tilde{f}_{\pm,\nu}\Big|_{\omega'} = \tilde{F}'_\pm(\nu)$. Then for all $\xi \in C^+$, with $[\xi \cdot b] \neq 0$, the form (82) of $\tilde{f}_{\pm,\nu}(\xi)$ follows from its homogeneity property, and this is sufficient for defining $\tilde{f}_{\pm,\nu}(\xi)$ as an $L^1-$function of $\xi$ on each cycle $\gamma$ equipped with the measure $\mathrm{d}\mu_\gamma$. We moreover see that the functions $\tilde{F}_\pm$ and $\tilde{F}'_\pm$ can be computed simply by fixing two special configurations of $\xi$, namely one has:

$$\tilde{F}_\pm(\nu) = \tilde{f}_{\pm,\nu}((1, 0, -1))$$

and

$$\tilde{F}'_\pm(\nu) = \tilde{f}_{\pm,\nu}((1, 0, 1)).$$

■

We shall now consider a system of $G_b-$invariant functions $\underline{\mathcal{W}}^-$, $\underline{\mathcal{W}}^+$, $\underline{\mathcal{C}}$, $\underline{\mathcal{R}}$, $\underline{\mathcal{A}}$, associated with an invariant perikernel W and compute the Lorentzian FH-transforms of these various functions. We shall see that it is sufficient to make the computation for the retarded function $\underline{\mathcal{R}}(x) = r(\cosh v)$: here,we have taken into account the support of $r$ by putting $x_2 = \cosh v$, with $v \geq 0$. In view of Proposition 14, we are led to compute the corresponding quantities $\tilde{F}_\pm(\nu) \equiv \tilde{\underline{\mathcal{R}}}_{\pm,\nu}((1,0,-1))$ and $\tilde{F}'_\pm(\nu) \equiv \tilde{\underline{\mathcal{R}}}_{\pm,\nu}((1,0,1))$.

To this purpose we introduce the following transforms $G$ and $H$ of $r$:

$$G(\nu) = \int_0^\infty Q_{-\frac{1}{2}+i\nu}(\cosh v)\ r(\cosh v)\ \sinh v\ \mathrm{d}v, \tag{83}$$

$$H(\nu) = \int_0^\infty P_{-\frac{1}{2}+i\nu}(\cosh v)\ r(\cosh v)\ \sinh v\ \mathrm{d}v. \tag{84}$$

In the latter $P_{-\frac{1}{2}+i\nu}$ and $Q_{-\frac{1}{2}+i\nu}$ denote respectively the first-kind and second -kind Legendre functions, and the well-known identity [18] $Q_{-\frac{1}{2}+i\nu} - Q_{-\frac{1}{2}-i\nu} = -i\pi \tanh \pi\nu\ P_{-\frac{1}{2}+i\nu}$ implies the following relation:

$$H(\nu) = H(-\nu) = \frac{G(\nu) - G(-\nu)}{-i\pi \tanh \pi\nu} \tag{85}$$

We can then show the following property:

**Proposition 15** *Under the assumption that $e^{\frac{v}{2}} r(\cosh v)$ is in $L^1(\mathbb{R}^+, \mathrm{d}v)$, the Lorentzian FH-transforms of $\underline{\mathcal{R}}$ are obtained by the following formulae:*

$$\tilde{\underline{\mathcal{R}}}_{\pm,\nu}(\xi) = |[\xi\cdot b]|^{-\frac{1}{2}-i\nu}\left(Y([\xi\cdot b])\tilde{F}(\nu) + Y(-[\xi\cdot b])\tilde{F}'_\pm(\nu)\right) \tag{86}$$

*where*

$$a) \qquad \tilde{F}(\nu) \equiv \tilde{\underline{\mathcal{R}}}_{\pm,\nu}((1,0,-1)) = G(\nu); \tag{87}$$

*$G(\nu)$ coincides* [4] *with the spherical Laplace transform [13] [11] of the Volterra kernel $\mathcal{R}(x,x')$ whose reduced form is $\underline{\mathcal{R}}$; $G(\nu)$ is the boundary value of a holomorphic function in the half-plane* $\mathrm{Im}\nu < 0$.

$$b) \ \ \tilde{F}'_\pm(\nu) \equiv \tilde{\underline{\mathcal{R}}}_{\pm,\nu}((1,0,1)) = \frac{\pi H(\nu)}{\cosh \pi\nu} \mp i e^{\pm\pi\nu} G(-\nu) \tag{88}$$

**Proof.** a) Let $\xi = (1,0,-1)$; since the support $\Gamma^+(b)$ of $\underline{\mathcal{R}}$ is contained in the region where $[x\cdot\xi] = x_0 + x_2$ is positive, we can put $[x\cdot\xi] = e^t$ and use the following parametrization of $\Gamma^+(b)$ in terms of $t$ and $v$:

$$\Gamma^+(b) \qquad \left\{\begin{array}{lll} x_0 & = & e^t - \cosh v \\ x_1 & = & \pm e^{t/2}\sqrt{2(\cosh t - \cosh v)} \\ x_2 & = & \cosh v \end{array}\right. \qquad \text{with } \ t \geq v \geq 0. \tag{89}$$

In these coordinates one has

$$\mathrm{d}\sigma(x) = \frac{e^{t/2}\sinh v\ \mathrm{d}t\ \mathrm{d}v}{2\sqrt{2(\cosh t - \cosh v)}} \tag{90}$$

We therefore have in view of (49) (since $(x_0+x_2)_\pm^{-\frac{1}{2}-i\nu} \equiv (x_0+x_2)^{-\frac{1}{2}-i\nu}$ in this region):

$$\tilde{F}(\nu) = \tilde{\underline{\mathcal{R}}}_{\pm,\nu}((1,0,-1)) = \int_{\Gamma^+(b)} (x_0+x_2)^{-\frac{1}{2}-i\nu} r(x_2)\ \mathrm{d}\sigma(x)$$

$$= 2\int_0^\infty r(\cosh v)\ \sinh v\ \mathrm{d}v \int_v^\infty \frac{e^{-i\nu t}\ \mathrm{d}t}{2\sqrt{2(\cosh t - \cosh v)}} = G(\nu); \tag{91}$$

[4] up to the change of variable $\nu \to \lambda = -\frac{1}{2} + i\nu$ and a normalization factor equal to 1/2: see e.g. Eqs (III.10),(III.11) of [11].

For obtaining the latter, we have taken into account our assumption on $r$ which ensures the convergence of the double integral and we have made use of the following integral representation of the second-kind Legendre function

$$Q_{-\frac{1}{2}+i\nu}(\cosh v) = \int_v^\infty \frac{e^{-i\nu t}\ \mathrm{d}t}{\sqrt{2(\cosh t - \cosh v)}} \tag{92}$$

We notice that in this configuration $\xi = (1, 0, -1)$, the two FH-transforms $\tilde{\underline{\mathcal{R}}}_{+,\nu}$ and $\tilde{\underline{\mathcal{R}}}_{-,\nu}$ of $\underline{\mathcal{R}}$ coincide (as functions of $\nu$) with the spherical Laplace transform of the Volterra kernel $\mathcal{R}$ (see our previous footnote and [13], [11] for a detailed study of this transform). Under our assumption on $r$, the integral in Eq. (91) is uniformly convergent for all complex $\nu$ such that $\mathrm{Im}\nu < 0$ and therefore extends the definition of $G(\nu)$ as a holomorphic function in this domain.

b) Let $\xi = (1, 0, 1)$; the support $\Gamma^+(b)$ of $\underline{\mathcal{R}}$ is now decomposed into two regions according to the sign of the scalar product $[x \cdot \xi] = x_0 - x_2$. The corresponding FH-transforms $\tilde{\underline{\mathcal{R}}}_{\pm,\nu}((1, 0, 1))$ are then given as the sum of two contributions

i) In the region (I)$= \Gamma^+(b) \cap \{x : x_0 - x_2 > 0\}$: we can put $[x \cdot \xi] = e^t$ and use the following parametrization of this region

$$\text{(I)} \quad \begin{cases} x_0 &= e^t + \cosh v \\ x_1 &= \pm e^{t/2}\sqrt{2(\cosh v + \cosh t)} \\ x_2 &= \cosh v \end{cases} \qquad \text{with } t \in \mathbb{R},\ \ v \geq 0. \tag{93}$$

In these coordinates one has

$$\mathrm{d}\sigma(x) = \frac{e^{t/2} \sinh v\ \mathrm{d}t\ \mathrm{d}v}{2\sqrt{2(\cosh v + \cosh t)}} \tag{94}$$

and therefore:

$$\int_{(\mathrm{I})} (x_0 - x_2)_\pm^{-\frac{1}{2}-i\nu} r(x_2) \mathrm{d}\sigma(x) = 2\int_0^\infty r(\cosh v) \sinh v\, \mathrm{d}v \int_{-\infty}^\infty \frac{e^{-i\nu t}\ \mathrm{d}t}{2\sqrt{2(\cosh v + \cosh t)}}$$

$$= \frac{\pi}{\cosh \pi\nu} \int_0^\infty P_{-\frac{1}{2}+i\nu}(\cosh v)\ r(\cosh v)\ \sinh v\, \mathrm{d}v = \frac{\pi}{\cosh \pi\nu} H(\nu). \tag{95}$$

ii) In the region (II) $= \Gamma^+(b) \cap \{x : x_0 - x_2 < 0\}$ we can put $[x \cdot \xi] = -e^t$, being careful that $(x_0 - x_2)_\pm^{-1/2-i\nu} = \mp i e^{\pm\pi\nu} e^{(-\frac{1}{2}-i\nu)t}$, and use the following parametrization of this region

$$\text{(II)} \quad \begin{cases} x_0 &= -e^t + \cosh v \\ x_1 &= \pm e^{t/2}\sqrt{2(\cosh t - \cosh v)} \\ x_2 &= \cosh v \end{cases} \quad \text{for } t \leq -v \ \ v \geq 0. \tag{96}$$

Since $\mathrm{d}\sigma(x)$ is again given by Eq. (90), the contribution of the region II is:

$$\int_{(\mathrm{II})} (x_0 - x_2)_\pm^{-\frac{1}{2}-i\nu}\ r(x_2) \mathrm{d}\sigma(x) =$$

$$\mp 2ie^{\pm\pi\nu} \int_0^\infty r(\cosh v) \sinh v\ \mathrm{d}v \int_{-\infty}^{-v} \frac{e^{-i\nu t} \mathrm{d}t}{2\sqrt{2(\cosh t - \cosh v)}}$$

$$= \mp i e^{\pm\pi\nu)} \int_0^\infty Q_{-\frac{1}{2}-i\nu}(\cosh v)\ r(\cosh v)\ \sinh v\ \mathrm{d}v = \mp i e^{\pm\pi\nu}\ G(-\nu). \tag{97}$$

Regrouping together the two contributions (95) and (97), we then obtain:

$$\tilde{\underline{\mathcal{R}}}_{\pm,\nu}((1, 0, 1)) = \frac{\pi H(\nu)}{\cosh \pi\nu} \mp i e^{\pm\pi\nu} G(-\nu), \tag{98}$$

which ends the proof of b). ■

Let us rewrite as follows the result of Proposition 15:

$$\underline{\tilde{\mathcal{R}}}_{\pm,\nu}(\xi) = |[\xi\cdot b]|^{-\frac{1}{2}-i\nu}\left\{Y([\xi\cdot b])G(\nu) + Y(-[\xi\cdot b])\left[\frac{\pi H(\nu)}{\cosh\pi\nu} \mp ie^{\pm\pi\nu}G(-\nu)\right]\right\} \tag{99}$$

By the same analysis, a similar formula can be obtained for the FH-transforms of the corresponding advanced function $\underline{\mathcal{A}}(x)$ :

$$\underline{\tilde{\mathcal{A}}}_{\pm,\nu}(\xi) = |[\xi\cdot b]|^{-\frac{1}{2}-i\nu}\times\cdots$$

$$\cdots\left\{\mp iY(-[\xi\cdot b])e^{\pm\pi\nu}G(\nu) + Y([\xi\cdot b])\left[\mp ie^{\pm\pi\nu}\frac{\pi H(\nu)}{\cosh\pi\nu} + G(-\nu)\right]\right\} \tag{100}$$

Taking into account Eqs (99) and (100), we now deduce the FH-transforms of the function $\underline{\mathcal{C}} = -i\,(\underline{\mathcal{R}} - \underline{\mathcal{A}})$ ; in view of (85), we obtain:

$$\underline{\tilde{\mathcal{R}}}_{\pm,\nu}(\xi) - \underline{\tilde{\mathcal{A}}}_{\pm,\nu}(\xi) = \pm i\,|[\xi\cdot b]|^{-\frac{1}{2}-i\nu}\,\pi H(\nu)\left\{e^{\pm i\pi(-\frac{1}{2}-i\nu)}Y(-[\xi\cdot b]) + Y([\xi\cdot b])\right\} \tag{101}$$

or

$$\underline{\tilde{\mathcal{C}}}_{\pm,\nu} = \pm\pi H(\nu)[\xi\cdot b]_{\pm}^{-\frac{1}{2}-i\nu}. \tag{102}$$

Since Proposition 8 can be applied to the holomorphic functions $\underline{\mathcal{W}}^{\pm}$, the FH-transforms of $\underline{\mathcal{C}} = \underline{\mathcal{W}}^- - \underline{\mathcal{W}}^+$ immediately yield those of $\underline{\mathcal{W}}^-$ and $\underline{\mathcal{W}}^+$, namely:

$$\underline{\tilde{\mathcal{W}}}^-_{+,\nu}(\xi) = \pi H(\nu)[\xi\cdot b]_+^{-\frac{1}{2}-i\nu}, \quad \underline{\tilde{\mathcal{W}}}^-_{-,\nu}(\xi) = 0, \tag{103}$$

$$\underline{\tilde{\mathcal{W}}}^+_{-,\nu}(\xi) = \pi H(\nu)[\xi\cdot b]_-^{-\frac{1}{2}-i\nu}, \quad \underline{\tilde{\mathcal{W}}}^+_{+,\nu}(\xi) = 0, \tag{104}$$

## 5.2 Connection between the inverse transformations

Let us restrict our attention to the function $\underline{\mathrm{W}}^-(z)$ and write the inversion formula (52) which expresses it in terms of its FH-transform $\underline{\tilde{\mathcal{W}}}^-_{+,\nu}(\xi) = \pi H(\nu)[\xi\cdot b]_+^{-\frac{1}{2}-i\nu}$. We obtain:

$$\underline{\mathrm{W}}^-(z) = \mathrm{w}([z\cdot b]) = \frac{1}{2\pi^2}\int_0^\infty \frac{\nu\tanh\pi\nu}{e^{\pi\nu}\cosh\pi\nu}\,\mathrm{d}\nu\int_\gamma [z\cdot\xi]^{-\frac{1}{2}+i\nu}\;\underline{\tilde{\mathcal{W}}}^-_{+,\nu}(\xi)\;\mathrm{d}\mu_\gamma(\xi)$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{\nu\tanh\pi\nu}{e^{\pi\nu}\cosh\pi\nu}\;H(\nu)\;\mathrm{d}\nu\int_\gamma [z\cdot\xi]^{-\frac{1}{2}+i\nu}[\xi\cdot b]_+^{-\frac{1}{2}-i\nu}\;\mathrm{d}\mu_\gamma(\xi) \tag{105}$$

But in the latter we recognize the integral representation (55) of the first-kind Legendre function, which therefore allows us to write:

$$\mathrm{w}([z\cdot b]) = \frac{1}{2}\int_0^\infty \frac{\nu\tanh\pi\nu}{\cosh\pi\nu}\;P_{-\frac{1}{2}-i\nu}([z\cdot b])\;H(\nu)\;\mathrm{d}\nu \tag{106}$$

Taking into account Eq.(85), we can at first transform the previous integral over $\nu$ into an integral on the full real axis (with a factor 1/2) and then purely replace $\tanh\pi\nu\;H(\nu)$ by $2\left(\frac{i}{\pi}\right)G(\nu)$ in the integrand since the remaining factor is an odd function of $\nu$. We thus obtain the following statement which is a special case of a general property established in [11], according to which a general perikernel with moderate growth (namely dominated by $|[z\cdot z']|^m$, with $m > -1$) can be decomposed linearly on the family of elementary perikernels $P_{m+i\nu}([z\cdot z'])$, with $\nu$ varying from $-\infty$ to $+\infty$.

**Theorem 3** *Let $W(z,z') = \mathrm{w}([z\cdot z'])$ be an invariant perikernel on $X$, whose associated retarded Volterra kernel $R$ satisfies the assumption of Proposition 15. Then it admits the following decomposition in terms of the spherical Laplace transform $G(\nu)$ of $R$:* [5]

$$\mathrm{w}([z\cdot z']) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\nu\, P_{-\frac{1}{2}-i\nu}([z\cdot z'])}{\cosh \pi\nu}\;\; G(\nu)\;\; \mathrm{d}\nu. \tag{107}$$

# 6 The chiral Fourier-Helgason transformation

## 6.1 Definition and properties of the transforms $\tilde{f}_{\rightarrow}$ and $\tilde{f}_{\leftarrow}$

Let us first show the following

**Proposition 16** *For every function $f$ in $H^2_{(reg)}(\mathcal{T}_{\rightarrow})$ or in $H^2_{(reg)}(\mathcal{T}_{\leftarrow})$), the corresponding Lorentzian FH-transforms $\tilde{f}_{+,\nu}(\xi)$ and $\tilde{f}_{-,\nu}(\xi)$ both vanish.*

**Proof.** It is sufficient to consider the case when $f$ belongs to $H^2_{(reg)}(\mathcal{T}_{\leftarrow})$: the corresponding function $\hat{f}(\lambda,\mu)$ is the boundary value of a function $\hat{F}(\lambda,\mu)$ holomorphic in the pierced tube $\tau^{+,+}\setminus\delta$ (see Proposition 7), which moreover satisfies the boundedness property of Definition 1. We then consider the following integral, similar to the one at the r.h.s. of Eq.(46), but taken on a cycle of the form $\mathbb{R}^2 + i(a\lambda_0, a\mu_0)$ with $a\geq 0,\ 0<\lambda_0<\mu_0$:

$$2^s\int_{\mathbb{R}^2+i(a\lambda_0,a\mu_0)}\left[\frac{\hat{F}(\lambda,\mu)}{\lambda-\mu}\right]\frac{[(\cos\alpha/2)\lambda-(\sin\alpha/2)]^s[(\cos\alpha/2)\mu-(\sin\alpha/2)]^s}{(\lambda-\mu)^{s+1}}\mathrm{d}\lambda\mathrm{d}\mu. \tag{108}$$

In the latter, each complex power of the form $t^s$ is fixed in its principal sheet (see subsection 4.1). In view of the analyticity and boundedness properties of the integrand, the latter integral is well-defined for $-1<\mathrm{Re}s<0$, and it is independent of $a$ (since the cycle remains in the tube $\tau^{+,+}\cap\{(\lambda,\mu);\ \mathrm{Im}(\lambda-\mu)<0\}$); moreover, it admits a bound which tends to zero when $a$ tends to infinity. Therefore it vanishes for all $a>0$. Now one checks that since the algebraic factor in the integrand is equal to $\left(\cos^2\frac{\alpha}{2}\right)^s(\lambda_\alpha-\mu_\alpha)^{-s}$, the limit of the integral (108) for $a$ tending to zero coincides with the r.h.s. of Eq. (47) *with the specification* $(\lambda_\alpha-\mu_\alpha)_-^{s+1}$. This shows that $\tilde{f}_+(\xi(\alpha),s)=0$ (for all values of $\alpha$). Choosing the previous cycle of integration in the tube $\tau^{+,+}\cap\{(\lambda,\mu);\ \mathrm{Im}(\lambda-\mu)>0\}$ (i.e. with $0<\mu_0<\lambda_0$), one shows similarly that $\tilde{f}_-(\xi,s)=0$. ■

In order to introduce the chiral FH-transformation, we need the following geometrical property of the complex cone $C^{(c)}=\{\xi=(\xi_0,\xi_1,\xi_2)\in\mathbb{C}^3;\ \xi_0^2-\xi_1^2-\xi_2^2=0\}$:

**Proposition 17** *All complex points $\xi\in C^{(c)}$ are such that $(\mathrm{Im}\xi)^2\leq 0$ and $C^{(c)}$ admits the following partition: $C^{(c)}=C\cup C_{\rightarrow}\cup C_{\leftarrow}$, where*

$$C_{\rightarrow}=\{\xi\in C^{(c)};\ \epsilon(\xi)=-\},$$

$$C_{\leftarrow}=\{\xi\in C^{(c)};\ \epsilon(\xi)=+\},$$

*with $\epsilon(\xi)=$ sgn $\mathrm{Det}(e,\mathrm{Re}\xi,\mathrm{Im}\xi)$ (see subsection 2.2).*

*The domains $C_{\rightarrow}$ and $C_{\leftarrow}$, (similar to the tuboids $\mathcal{T}_{\rightarrow}$ and $\mathcal{T}_{\leftarrow}$ of $X^{(c)}$) are determined by their bases in the complex circle $\gamma_0^{(c)}$ of $C^{(c)}$, parametrized by two half-planes:*

$$(\gamma_0^{(c)})_{\rightarrow}=\{\xi=\xi(\Phi)=(1,\sin\Phi,\cos\Phi);\ \Phi=\phi+i\eta,\ \eta>0\}$$

$$(\gamma_0^{(c)})_{\leftarrow}=\{\xi=\xi(\Phi)=(1,\sin\Phi,\cos\Phi);\ \Phi=\phi+i\eta,\ \eta<0\}$$

[5] In the context of de Sitter quantum field theory, this result has been interpreted in [6] as a Källen-Lehmann-type representation for the two-point functions of general interacting fields.

In the same spirit as for the Lorentzian FH-transformations (see proposition 8 and Definition 2), one could introduce the chiral transformations by specifying boundary value prescriptions $[x_{\rightarrow} \cdot \xi]^s$ and $[x_{\leftarrow} \cdot \xi]^s$ for the FH-kernel, corresponding to limits of the holomorphic function $[z \cdot \xi]^s$ from the respective tuboids $\mathcal{T}_{\rightarrow}$ and $\mathcal{T}_{\leftarrow}$ of $X^{(c)}$ and put correspondingly:

$$\tilde{f}_{\leftrightarrows}(\xi, s) = \int_X [x_{\leftrightarrows} \cdot \xi]^s f(x) \; \mathrm{d}\sigma(x).$$

However, two new features appear in this case:

1) The non-uniformity of the function $[z \cdot \xi]^s$ for general $s$ leads one to define the transform *only for $s$ integer and negative* (for the sake of convergence). (Note that in the $(\lambda, \mu)-$ representation of $X^{(c)}$ (see section 3), this non-uniformity is easily seen to be due (see Eq. (29)) to the non-trivial homotopy of the pierced tubes $\tau^{\pm,\pm} \setminus \delta$ which represent $\mathcal{T}_{\leftrightarrows}$).

2) Instead of considering $x$ as the limit of points in $X^{(c)}$, one can equivalently in this situation consider $\xi$ as the limit of points in either one of the domains $C_{\rightarrow}$, $C_{\leftarrow}$ of $C^{(c)}$. The advantage of the latter is the derivation, as a by-product, of the analyticity with respect to $\xi$ in the domains $C_{\leftrightarrows}$ of the transforms $\tilde{f}_{\leftrightarrows,\ell}(\xi)$ thus obtained.

It is convenient to use here the parametrization $z = z[\theta, \Psi]$ (see Eq. (18)) of $X^{(c)}$ which, for $\xi \in \gamma_0^{(c)}$, yields the following expression of $[z \cdot \xi]$:

$$[z[\theta, \Psi] \cdot \xi(\phi + i\eta)] = \sinh \Psi - \cosh \Psi \cos(\theta - \phi - i\eta). \tag{109}$$

As shown by this expression, for any fixed complex values of $\xi$ ($\xi \in C_{\leftrightarrows}$), $\Psi$ and $\mathrm{Im}\theta$, the complex point $[z[\theta, \Psi] \cdot \xi(\phi + i\eta)]$ varies on an ellipse parametrized by $\mathrm{Re}\theta$. For $z = x \in X$ (i.e. $\Psi$ and $\theta$ real), this ellipse encloses the origin; this shows the necessity of defining the FH-kernel with $s$ integer. These considerations lead one to the following

**Proposition-Definition 3** *Given a function $f$ in $\mathcal{H}_{(reg)}(X)$, we define its* chiral Fourier-Helgason transforms *as the following two sequences of functions:* $\{\tilde{f}_{\leftrightarrows,\ell}(\xi),\ \ell$ integer $\geq 0\}$ *:*

$$\tilde{f}_{\leftrightarrows,\ell}(\xi) = \int_X [x \cdot \xi]^{-\ell-1} f(x) \; \mathrm{d}\sigma(x); \tag{110}$$

*for each $\ell$, the two functions $\tilde{f}_{\rightarrow,\ell}(\xi)$ and $\tilde{f}_{\leftarrow,\ell}(\xi)$ are defined by the integral at the r.h.s. as holomorphic functions in the respective domains $C_{\rightarrow}$ and $C_{\leftarrow}$ of $C^{(c)}$.*

*Moreover, for every function $f \in H^2_{(reg)}(\mathcal{T}_{\rightarrow})$ (resp. $H^2_{(reg)}(\mathcal{T}_{\leftarrow})$), there is a* unique *(non-vanishing) chiral FH-transform which is $\{\tilde{f}_{\rightarrow,\ell}(\xi)\}$ (resp. $\{\tilde{f}_{\leftarrow,\ell}(\xi)\}$).*

The proof of the latter statement, namely the fact that the integral (110) vanishes e.g. for $\xi \in C_{\rightarrow}$ if $f$ belongs to $H^2_{(reg)}(\mathcal{T}_{\leftarrow})$, is proved by contour-distortion in the complex $\theta-$plane (making use of the parametrization (18) of $X^{(c)}$ and of (109)): this is an exact counterpart of the result of Proposition 8 for the Lorentzian case. Similarly, one would also prove that the chiral FH-transforms both vanish for all the functions $f$ which belong to $H^2_{(reg)}(\mathcal{T}^{\pm})$ (i.e. the analog of Proposition 16).

## 6.2 Inversion of the transformation

In order to prove the analog of Theorem 2 for the case of the chiral FH-transformation, we need to define for each point $z$ in $\mathcal{T}_{\rightarrow}$ (resp. $\mathcal{T}_{\leftarrow}$) an appropriate class of relative cycles $\gamma(z)$ in $H^1(C^{(c)}, \{\xi;\ [z \cdot \xi] = 0\})$ with support contained in $C_{\rightarrow}$ (resp. $C_{\leftarrow}$): the end-points of this support will respectively belong to the two linear generatrices of the cone $C^{(c)}$ defined by the equation $[z \cdot \xi] = 0$. In view of Proposition 5, it is sufficient to define $\gamma(z)$ for $z \in h_{\rightarrow}$ (resp. $h_{\leftarrow}$), i.e. of the form $z = z_v = (0, i \sinh v, \cosh v)$, $v > 0$ (resp. $v < 0$). In that case, we specify the cycle $\gamma(z_v)$ in the manifold $\gamma_0^{(c)} = \{\xi = \xi(\Phi) = (1, \sin \Phi, \cos \Phi);\ \Phi = \phi + i\eta\}$ as follows: since $[z \cdot \xi] = -\cos(\Phi - iv)$ vanishes at $\Phi = \pm\frac{\pi}{2} + iv$, we choose $\gamma(z_v)$ as the path

$\phi \to \Phi = \phi + iv$, with $\phi$ increasing from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$: according to the sign of $v$, the support of $\gamma(z_v)$ belongs either to $C_\to$ or to $C_\leftarrow$. For an arbitrary point $z$ in $\mathcal{T}_\to$ (resp. $\mathcal{T}_\leftarrow$), which is obtained (in view of proposition 5) by the action of a certain transformation $g$ of $G$ on an appropriate point $z_v$, the corresponding cycle $\gamma(z)$ is defined by the action of $g$ on $\gamma(z_v)$, its support being contained in the corresponding one-dimensional complex manifold $\gamma^{(c)} = g\gamma_0^{(c)}$. This is satisfactory since the domains $C_\to$ and $C_\leftarrow$ as well as the set with equation $[z\cdot\xi] = 0$ are invariant under the action of $G$. Moreover, by considering any path in the group $G$ as a union of arbitrarily small successive paths, one easily sees that in the above construction the cycle $\gamma(z)$ remains for all $z$ in a continuously varying relative homology class in $H^1(C^{(c)}, \{\xi;\ [z\cdot\xi] = 0\})$ [17] (since the support of $\gamma(z)$ can be distorted by a succession of small homotopies keeping its end-points respectively in the generatrices of $C^{(c)}$ with equation $[z\cdot\xi] = 0$).

The following geometrical property will play an important role:

**Proposition 18** *Let $z \in \mathcal{T}_\to$ and $z'$ in the closure of $\mathcal{T}_\leftarrow$. Then for all $\xi \in$ supp $\gamma(z)$ (with the above definition of $\gamma(z)$), one has $[z'\cdot\xi] \neq 0$.*

**Proof.** Using again the $G-$invariance properties of $\mathcal{T}_\leftrightarrows$, we can take $z = z_v$ and $\xi \in \gamma(z_v)$, while $z'$ is represented as follows (in view of (18) and (21)):

$$z' = (\sinh(\psi + i\varphi),\ \cosh(\psi + i\varphi)\sin(u + iv'),\ \cosh(\psi + i\varphi)\cos(u + iv'))$$

with $\tanh v' \leq -\frac{|\sin\varphi|}{\cosh\psi}$.

We then have (since $v > 0$): $\tanh(v' - v) < -\frac{|\sin\varphi|}{\cosh\psi}$, which implies:

$$\cosh(v - v') > \frac{\cosh\psi}{(\cosh^2\psi - \sin^2\varphi)^{\frac{1}{2}}} \quad \text{or} \quad e^\psi + e^{-\psi} < 2\cosh(v - v')|\cosh(\psi + i\varphi)|. \tag{111}$$

On the other hand, one has:

$$[z'\cdot\xi] = \sinh(\psi + i\varphi) - \cosh(\psi + i\varphi)\cos(u - \phi + i(v' - v)), \qquad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2},$$

which shows that the complex point $[z'\cdot\xi]$ varies on an ellipse whose major axis is equal to $2a = 2\cosh(v - v')|\cosh(\psi + i\varphi)|$ and whose foci are the points $F = e^{\psi + i\varphi}$ and $F' = e^{-(\psi + i\varphi)}$. Then the inequality (111) means that $OF + OF' < 2a$, namely that the origin is always strictly inside the ellipse described by the point $[z'\cdot\xi]$ and therefore the statement $[z'\cdot\xi] \neq 0$ is proved . ■

Using the relative cycles $\gamma(z)$ previously defined, we can now prove the following analog of Theorem 2:

**Theorem 4** *Let $f(x)$ belong to $H^2_{(reg)}(\mathcal{T}_\leftrightarrows)$ and let $\tilde{f}_{\leftrightarrows,\ell}(\xi)$ be its chiral FH-transform. Then the holomorphic function $F(z)$ in $\mathcal{T}_\leftrightarrows$ whose boundary value is $f$ is given in terms of $\tilde{f}_{\leftrightarrows,\ell}$ by the following formula:*

$$F(z) = \frac{1}{4\pi^2}\sum_{\ell=0}^{\infty}(2\ell + 1)\int_{\gamma(z)}[z\cdot\xi]^\ell\ \tilde{f}_{\leftrightarrows,\ell}(\xi)\ \mathrm{d}\mu_{\gamma^{(c)}}(\xi) \tag{112}$$

*In the latter, $\mathrm{d}\mu_{\gamma^{(c)}}$ is the restriction to $\gamma^{(c)}$ of the holomorphic extension of the form $[i_\Xi\omega](\xi)$ (see 4.2) to $C^{(c)}$.*

One notices that the integral over $\xi$ at the r.h.s. of Eq. (112) is well-defined and holomorphic in $\mathcal{T}_\leftrightarrows$ since the function $\tilde{f}_{\leftrightarrows,\ell}(\xi)$ is holomorphic in the corresponding domain $C_\leftrightarrows$ of $C^{(c)}$ and therefore integrable on $\gamma(z)$, by construction of the latter. As a matter of fact, in view of this analyticity property, the corresponding integral on the real cycle $\gamma$ (used for the Lorentzian inversion formula (52)) would give a vanishing integral in the present case.

In order to prove Theorem 4, one makes use of the Cauchy-type representation (38) for the case of the tubes $\mathcal{T}_\leftrightarrows$ together with the following expression of the Cauchy kernel on $X^{(c)}$ whose proof is given below:

**Proposition 19** *The Cauchy kernel on $X^{(c)}$ is given by the following double sum-integral*

$$\frac{1}{(z'-z)^2} = \frac{1}{4}\sum_{\ell=0}^{\infty}(2\ell+1)\int_{\gamma(z)}[z\cdot\xi]^{\ell}\ [\xi\cdot z']^{-\ell-1}\mathrm{d}\mu_{\gamma^{(c)}}(\xi) \tag{113}$$

*which is absolutely convergent for $(z,z')$ in $\mathcal{T}_{\rightarrow}\times\mathcal{T}_{\leftarrow}$ and in $\mathcal{T}_{\leftarrow}\times\mathcal{T}_{\rightarrow}$. This formula remains meaningful when one of the points is taken real, e.g. $z'=x$, provided the limit is taken from the appropriate tuboid.*

If we plug the expression (113) of $\left[(x-z)^2\right]^{-1}$ into the r.h.s. of (38), invert the integrals over $x$ and over $(\ell,\xi)$ and take into account the defining formula (110) of $\tilde{f}_{\leftrightarrows,\ell}(\xi)$, we readily obtain formula (112) for both types of configurations . So, proving Theorem 4 amounts to proving Proposition 19.

### 6.3 Chiral invariant perikernels: a new representation for the second-kind Legendre functions and the Cauchy kernel on $X^{(c)}$

We now introduce a class of $G^{(c)}-$ invariant kernels on $X^{(c)}$ which play the same role as the perikernels of subsection 4.3 with respect to the chiral tuboids. [6]

Such a "chiral invariant perikernel" is a holomorphic function $\mathrm{W}(z,z')$ holomorphic in the "cut-domain" $\Delta_{\chi}=\{(z,z')\in X^{(c)}\times X^{(c)};\ [z\cdot z']\in\Theta_{\chi}=\mathbb{C}\setminus[-1,+1]\}$ and satisfying (54). It then follows from Proposition 6 that W is holomorphic in particular in the two tuboids $\mathcal{T}_{\rightarrow}\times\mathcal{T}_{\leftarrow}$ and $\mathcal{T}_{\leftarrow}\times\mathcal{T}_{\rightarrow}$ of $X^{(c)}\times X^{(c)}$. In view of its $G-$invariance property (54), $\mathrm{W}(z,z')$ can also be identified with a function (i.e. its reduced form) $\mathrm{w}([z\cdot z'])$ holomorphic in the cut-plane $\Theta_{\chi}$.

This provides a second class of Lorentz invariant kernels $\mathcal{W}(x,x')$ on $X$, obtained as the boundary value of a holomorphic function $\mathrm{W}(z,z')$ from the analyticity domain $\mathcal{T}_{\leftarrow}\times\mathcal{T}_{\rightarrow}$ (or its opposite).

Basic examples of such chiral invariant perikernels [7] are provided by the second-kind Legendre functions, as shown by the following statement

**Proposition 20** *The following integral representation holds for $(z,z')$ varying in the pair of domains $\mathcal{T}^{r,l}=\mathcal{T}_{\rightarrow}\times\mathcal{T}_{\leftarrow}$ and $\mathcal{T}^{l,r}=\mathcal{T}_{\leftarrow}\times\mathcal{T}_{\rightarrow}$:*

$$Q_{\ell}([z\cdot z']) = (-1)^{\ell+1}\frac{1}{2}\int_{\gamma(z)}[z\cdot\xi]^{\ell}\ [\xi\cdot z']^{-\ell-1}\mathrm{d}\mu_{\gamma^{(c)}}(\xi) \tag{114}$$

*The latter integral is still convergent when one of the points, e.g. $z'$, is taken as a limiting real point from either tuboid $\mathcal{T}_{\rightarrow}$ or $\mathcal{T}_{\leftarrow}$, $z$ remaining inside the opposite tuboid. It moreover defines the second-kind Legendre functions $Q_{\ell}$ as a family of chiral invariant perikernels depending on the integer $\ell$.*

**Proof.** The independence of the integral (114) with respect to the choice of the cycle $\gamma(z)$ inside its homology class is a consequence of proposition 10 and of Stokes theorem, since $\gamma(z)$ is a relative cycle whose boundary belongs to an analytic set [17], namely the set with equation $[z\cdot\xi]=0$. Now the fact that this integral defines a holomorphic function of $(z,z')$ in the product tuboid $\mathcal{T}^{r,l}$ (and in its opposite) including also the limiting configurations with $z'$ real (as described in the statement) is a direct consequence of Proposition 18, since in that domain the factor $[\xi\cdot z']^{-\ell-1}$ never becomes singular for $\xi$ varying in $\gamma(z)$. Moreover, since the chosen representatives $\gamma(z)$ are precisely distorted by the action of the group $G$ on $\gamma(z_v)$ when $z$ varies in its tuboid, this implies the invariance property of the integral (114) under all the transformations $(z,z')\rightarrow(gz,gz')$; $g\in G$ (accompanied by the allowed change $\gamma(z)\rightarrow g\gamma(z)$). The function defined by this integral is therefore a function of the invariant $[z\cdot z']$, holomorphic in the image $\Theta_{\chi}$ of $\mathcal{T}^{r,l}$

[6] In dimension $d$, ($d\geq 2$), such kernels arise in the context of quantum field theory on a $d-$dimensional quadric of signature $(+,+,-,\cdots,-)$, interpreted as a $d-$dimensional anti-de Sitter spacetime manifold.

[7] In the application to anti-de Sitter quantum field theory, these examples are interpreted as the two-point functions (or "propagators") of massive Klein-Gordon-like fields [8] up to an appropriate normalization.

Let $\mathrm{W}_{(\ell)}^{r,l}(z,z')$ be the holomorphic function defined by the integral (114) in the tuboid $\mathcal{T}^{r,l}$. In order to compute $\mathrm{W}_{(\ell)}^{r,l}(z,z') = \mathrm{w}_{(\ell)}^{r,l}([z\cdot z'])$ explicitly, we choose a simple configuration in the border of $\mathcal{T}^{r,l}$, namely the pair $(z,z') = (z_v, b)$, which is such that $[z\cdot z'] = -\cosh v$. The corresponding cycle $\gamma(z_v)$ in the manifold $\gamma_0^{(c)}$ has been defined at the beginning of subsection 6.2. We then obtain (by going successively from the integration variable $\phi$ which parametrizes $\gamma(z_v)$ to $t = \tan\phi$):

$$\mathrm{w}_{(\ell)}^{r,l}([z\cdot z']) = (-1)^{\ell+1}\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{-(\ell+1)}(\phi + iv)\ \cos^{\ell}\phi\ \mathrm{d}\phi$$

$$= (-1)^{\ell+1}\frac{1}{2}\int_{-\infty}^{\infty}(\cosh v - it\sinh v)^{-(\ell+1)}\frac{\mathrm{d}t}{(1+t^2)^{\frac{1}{2}}} \tag{115}$$

By contour distortion to $t = i\tau;\ \tau \geq 1$ and putting $\tau = \cosh u$, one gets

$$\mathrm{w}_{(\ell)}^{r,l}([z\cdot z']) = (-1)^{\ell+1}\int_{0}^{\infty}(\cosh v + \cosh u \sinh v)^{-(\ell+1)}\ \mathrm{d}u = Q_\ell([z\cdot z']) \tag{116}$$

A similar computation would yield the same result by taking configurations $(z,z')$ in the opposite tuboid $\mathcal{T}^{l,r}$. ■

**Remark 6** *The function* $\mathrm{W}_{(\ell)}(z,z')$ *is a solution in both variables of the Laplace-Beltrami equation* $[\Delta_X + \ell(\ell+1)]\mathrm{W}_{(\ell)}(z,z') = 0$ *on the manifold* $X^{(c)}$ *(or in other words, the Klein-Gordon equation on complexified anti-de Sitter spacetime). For such a* $G-$*invariant solution, this equation reduces to a Legendre equation in the variable* $[z\cdot z']$ *and the required analyticity domain* $\Theta_\chi = \mathbb{C}\setminus[-1,+1]$ *selects the second-kind Legendre function* $Q_\ell([z\cdot z'])$ *as the unique relevant solution (up to a constant factor).*

We now come back to the Cauchy kernel $[(z-z')^2]^{-1} = \{-2([z\cdot z']+1)\}^{-1}$, whose analyticity in $\Theta_\chi$ and $G-$invariance show that it also defines a *chiral* invariant perikernel on $X^{(c)}$: we prove that the r.h.s. of Eq. (113) is precisely a representation of this perikernel in the pair of tuboids $\mathcal{T}^{r,l}$ and $\mathcal{T}^{l,r}$.

**Proof of Proposition 19**

As in the case of Proposition 11, one could give a complete computation exhibiting a Cauchy-Fantappié form of the kernel (proof B)), but for brevity we shall only give the fast proof, based on the previous integral representation (114) of the second-kind Legendre functions and on the following well-known formula [18], valid for all $Z \in \Theta_\chi$:

$$\frac{1}{Z-1} = \sum_{\ell=0}^{\infty}(2\ell+1)Q_\ell(Z) \tag{117}$$

By plugging the expression (114) of $Q_\ell(-[z\cdot z']) = (-1)^{\ell+1}Q_\ell([z\cdot z'])$ into the r.h.s. of (117) and taking into account the integrability conditions specified in Proposition 20, one readily obtains (113) with the specifications of Proposition 19.

# 7 Conclusion: Fourier-Helgason representation of the holomorphic decomposition in the tuboids

Being given any function $f(x)$ in $\mathcal{H}_X$, it admits a decomposition of the form $f = f^+ + f^- + f_\rightarrow + f_\leftarrow$, where each function is the boundary value on $X$ of a holomorphic function in the corresponding tuboid $\mathcal{T}^+, \mathcal{T}^-, \mathcal{T}_\rightarrow, \mathcal{T}_\leftarrow$ of $X^{(c)}$. This decomposition of $f$ has been obtained in section 3 by applying to $f$ a Cauchy kernel acting as a projection operator onto the Hardy space of each of the four tuboids. The results of section 4, in particular Proposition 8 and Theorem 2, completed by Proposition 16, show that in this decomposition the first two components $f^+$ and $f^-$ are completely characterized respectively by

the Lorentzian Fourier-Helgason transforms $\tilde{f}_{-,\nu}(\xi)$ and $\tilde{f}_{+,\nu}(\xi)$ of $f$. Each of these transforms lives on $C^+_{(\xi)} \times \mathbb{R}_{+(\nu)}$, where $\nu$ labels the principal series of irreducible representations of $SO_0(1,2)$. Formulae (49) and (52) then imply the following *Plancherel formulae* which introduce $L^2-$isomorphisms between the Hardy spaces $H^2(\mathcal{T}^{\pm})$ and the appropriate spaces of FH-transforms on $C^+_{(\xi)} \times \mathbb{R}_{+(\nu)}$ (note that equivalent formulae making use of the range $\nu \in ]-\infty, 0]$ instead of $[0, +\infty[$ for the FH-transforms are also valid) .

For any pair of functions $(f, g)$ in $\mathcal{H}(X)$ and their corresponding decompositions, one has:

$$\int_X \overline{f^-(x)}\ g^-(x)\ \mathrm{d}\sigma(x) = \frac{1}{2\pi^2} \int_0^\infty \frac{\nu \tanh \pi\nu}{e^{\pi\nu} \cosh \pi\nu} \mathrm{d}\nu \int_\gamma \overline{\tilde{f}_{+,\nu}(\xi)}\ \tilde{g}_{+,\nu}(\xi)\ \mathrm{d}\mu_\gamma(\xi) \tag{118}$$

and

$$\int_X \overline{f^+(x)}\ g^+(x)\ \mathrm{d}\sigma(x) = \frac{1}{2\pi^2} \int_0^\infty \frac{\nu \tanh \pi\nu}{e^{-\pi\nu} \cosh \pi\nu} \mathrm{d}\nu \int_\gamma \overline{\tilde{f}_{-,\nu}(\xi)}\ \tilde{g}_{-,\nu}(\xi)\ \mathrm{d}\mu_\gamma(\xi) \tag{119}$$

Similarly, the results of section 6 (Proposition-Definition 3 and Theorem 4) show that the other two components $f_\rightarrow$ and $f_\leftarrow$ of the decomposition of $f$ are completely characterized respectively by the chiral Fourier-Helgason transforms $\tilde{f}_{\rightarrow,\ell}(\xi)$ and $\tilde{f}_{\leftarrow,\ell}(\xi)$ of $f$. These transforms live respectively on $C_\rightarrow \times \mathbb{N}$ and $C_\leftarrow \times \mathbb{N}$ and are associated with the discrete series of irreducible representations of $SO_0(1,2)$. Corresponding Plancherel formulae are expected to follow from formulae (110) and (112).

In conclusion, we have given for the functions on the one-sheeted hyperboloid an explicit treatment of the Gelfand-Gindikin program, exhibiting in that case the construction of a holomorphic decomposition into four tuboids and its identity with an appropriately defined Fourier-Helgason decomposition in terms of the range of the corresponding irreducible representations of $G$. Although more sophisticated, the result remains very close to the holomorphic decomposition into four tubes and the corresponding support decomposition in the Fourier variables for the functions on $\mathbb{R}^2$.

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